

Learning to read, and learning about recurrence relations
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An excerpt from the text (almost).

Definition 1. A recurrence relation is called Halloween-like of degree k if it is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

Theorem 1. Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2 . Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants.

Theorem 2. Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 . Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants.

- (1) Are there any mathematical terms that are used for which you do not know the definition? List them.
- (2) Notice that there are two different fonts used in the definition. What is the significance of this?
- (3) Notice that both Greek and English letters are used in the theorem. Circle the Greek letters to make sure you don't mix them up with the English letters.
- (4) For each of the sequences defined below, identify the recurrence relation versus the initial conditions. Then figure out if the recurrence relation is Halloween-like by comparing it to the definition.
 - (a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 8$
 - (b) $a_n = 5a_{n-1} - 6a_{n-3}$ for $n \geq 3$, $a_0 = 3$, $a_1 = 8$, $a_2 = 10$
 - (c) $a_n = 2na_{n-1} + a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
 - (d) $a_n = a_{n-1} + 2$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
 - (e) $a_n = a_{n-1}^2 + a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
 - (f) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
- (5) For those recurrence relations that are Halloween-like, identify k and c_1, c_2, \dots, c_k by comparing the relation with the definition once again.
- (6) Which of your Halloween-like recurrence relations are of degree 2?
- (7) For each of the recurrence relations which are Halloween-like of degree 2, answer the following questions:
 - (a) Write down the polynomial $r^2 - c_1 r - c_2$ using the values of c_1 and c_2 you have found.
 - (b) Figure out which of the two theorems applies.
 - (c) Use the appropriate theorem to find a closed formula for a_n that involves α_1 and α_2 . (Remember: α_1 is totally different from a_1 , for example: the Greek letters and the English letters represent different things.)
 - (d) Use the initial conditions to find α_1 and α_2 and write down your final closed formula for a_n .

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- (e) Check that your closed formula satisfies the initial conditions.
 - (f) Check that your closed formula satisfies the recurrence relation by starting with the right hand side of the recurrence relation and seeing if it simplifies to a_n when you use your closed formula.
- (8) The actual term is not “Halloween-like of degree k ” but rather ”linear homogeneous of degree k with constant coefficient”. Cross out “Halloween-like of degree k ” and replace it with the actual term. At this stage we have to think of this definition as the definition of one mathematical concept since the individual words (“linear”, ”homogeneous”, “constant coefficients”) are not being defined separately. Each of these words does in fact have a meaning (that is not given here). The next series of questions invite you to make a guess about what the individual words might mean on their own.
- (9) Which of the given recurrence relations does not have constant coefficients?
 - (10) Which of the given recurrence relations is not linear?
 - (11) By process of elimination, can you guess which of the given recurrence relations is not homogeneous?