Languages and Group Theory, CSC-161, Dr. Ostheimer

1. Introduction

Learning Objectives:

(1) Get to know each other.

(2) Practice working with each other to the benefit of all.

(3) Practice close reading.

(4) Practice analytical thinking.

(5) Review an important topic from CS24.

(6) Start to understand what a theoretical computer scientist means by a “language”.

Group Formation:
You will be in groups of two or three. Each group will hand me **one** sheet of paper at the end of class. This does not count toward your grade.

Start by choosing roles. These are designed to help you work together effectively.

- the *scribe*: the person who writes down the conclusions of the entire group. The full names of each member of the group should be written on the paper. It is important that the scribe only writes down stuff that everyone in the group understands. If there is unresolved dispute within the group, the scribe writes that down.
- the *collaborator*: the person who finds a collaborator from another group (or the professor) who can help them. By the end of the day each collaborator should have met at least two other collaborators. Make sure the scribe writes down the names of collaborators who helped.
- the *facilitator*: the person who makes sure that every voice is heard equally. If there are only two people in your group, the collaborator also acts as facilitator. This is a good job for someone who is comfortable asking people questions about what they are thinking. At the end of.the session you should all discuss how effectively you worked together, and share any ideas or questions about how you could improve that. Add a comment or question for me about this to the sheet that you hand me.

2. New Definitions

Here are some definitions you will work with today. I recommend you **wait** to read them until you need them as you go through the questions in the next section.

Definition 1. *An* alphabet *is a nonempty finite set of symbols.*

Definition 2. *A* word *over an alphabet A is any string of symbols from A, including the* empty word, Λ, *the string with no symbols.*

Definition 3. *Let A be an alphabet. The* Kleene closure A^* *of A is the set of all words over A.*

Definition 4. *A language over an alphabet A is a subset of A^*.*

Definition 5. *A language L over alphabet A is* closed under concatenation *if for all $u, v \in L$, the concatenation uv is also an element of L.*

3. Your Questions

For the duration of this group work, we will let $X = \{a, b, A, B\}$.

Question 1. True *or* **False**: *$X = \{a, b, A, B\}$ is an alphabet. (Hint: read the definition of* alphabet*.)*

Question 2. True or False: *$abbaa$ is a word over X. (Hint: read the definition of* word*.)*

Now consider the $L_1 = X^*$.

Question 3. True or False: *L_1 is a language. (Hint: you know what to read.)*

Question 4. True or False: $X \subseteq L_1$

Question 5. True or False: $L_1 \subseteq X$

Question 6. True or False: *L_1 is closed under concatenation. (Hint: read?)*

Question 7. *Enumerate all the words in L_1 of length 2 or less in length-plus-lexicographic order. In this order, shorter words precede longer words, and words of equal length are listed in alphabetical order. Assume that $a < A < b < B$.*

Question 8. *How many words are in L_1?*

We will now define a relation R on L_1 using the following "rewriting rules":

$$\begin{aligned} aA &\rightarrow \Lambda \\ Aa &\rightarrow \Lambda \\ bB &\rightarrow \Lambda \\ Bb &\rightarrow \Lambda \end{aligned}$$

We define relation R to be the set of pairs $(w_1, w_2) \in L_1 \times L_1$ such that w_2 can be obtained from w_1 by using the rewriting rules (as many times as you want) in either direction. For example, $(baABa, aAbBa) \in R$ since

$$baABa \rightarrow bBa \rightarrow a \rightarrow bBa \rightarrow aAbBa.$$

Question 9. True or False: $(aabb, abab) \in R$.

Question 10. True or False: *R is indeed a relation on L_1. (Hint: read the definition of* relation *in the last section of the last handout.)*

Question 11. True or False: *R is reflexive. (Hint: you know what I am going to say.)*

Question 12. **True or False:** *R is symmetric.*

Question 13. **True or False:** *R is transitive.*

Question 14. **True or False:** *R is an equivalence relation.*

Question 15. *Give three examples of words in the equivalence class of $baABa$ with respect to R.*

Question 16. *Give three examples of words in L_1 that are not in the equivalence class of $baABa$.*

Question 17. *How many equivalence classes does R have?*

Question 18. *How many words are in each equivalence class?*

Let L_2 be the set of words in L_1 that are equivalent to Λ.

Question 19. **True or False:** *L_2 is a language over X.*

Question 20. *Find 5 words in L_2. Try to pick words that illustrate the range of different kinds of words that are in L_2.*

Question 21. *Find 5 words in L_1 that are not in L_2.*

Question 22. **True or False:**

- *(a) $X \subseteq L_2$;*
- *(b) $L_1 \subseteq L_2$;*
- *(c) $L_2 \subseteq L_1$;*
- *(d) L_2 is closed under concatenation.*

Question 23. **optional problem to be done outside of class with friends**

- *Describe an algorithm for deciding whether a given word in X^* is in L_2.*
- *Analyze the time complexity of your algorithm. In other words, find a big-O estimate for a function $f(n)$ that is the maximum number of steps needed to complete your algorithm if the length of the given word is n.*

4. Optional Extra Credit Problems

In my classes, extra credit problems are designed for students who are finding the material in class easy and are looking for more stimulating work. They are not designed for students trying to improve their grades. That's what homework is for.

This is a problem from my own area of research which is on the boundary of theoretical computer science and group theory, a branch of theoretical computer science which is introduced in Math 145. We consider the more general case in which we are given a finite set of rewriting rules that *includes* the rewriting rules given above, but which *may not be* limited to those.

(1) Consider three examples of rewriting rules:

- the example considered above;
- the example obtained by adding the relation $ab \to ba$;
- the example obtained by adding, furthermore, the relations $a^3 \to \Lambda$ and $b^2 \to \Lambda$.

For each example, answer the following questions:

(a) How many equivalence classes are there?

(b) Describe a set of shortest representatives for each equivalence class using length-plus-lexicographic order.

(c) Consider the language of words which are equivalent to Λ. Describe an algorithm for deciding whether a given word in X^* is in this language.

(d) Analyze the time complexity of your algorithm.

5. Equivalence Relations

Definition 6. *Let A be a non-empty set. A* relation *on A is a subset of $A \times A$.*

Definition 7. *Let R be a relation on set A. R is* reflexive *if for all $a \in A$, $(a, a) \in R$.*

Definition 8. *Let R be a relation on set A. R is* symmetric *if for all $a, b \in A$,*

$$(a, b) \in R \to (b, a) \in R$$

Definition 9. *Let R be a relation on set A. R is* transitive *if for all $a, b, c \in A$,*

$$(((a, b) \in R) \wedge ((b, c) \in R)) \to ((a, c) \in R)$$

Definition 10. *Let R be a relation on set A. R is an* equivalence relation *if it is reflexive, symmetric and transitive.*

Definition 11. *Let R be an equivalence relation on A. If $a, b \in A$ and $(a, b) \in R$, we say that a is* equivalent to b with respect to R.

Definition 12. *Let R be an equivalence elation on A. If $a \in A$ then the* equivalent class *$[a]_R$ is the set of all words $b \in A$ such that b is equivalent to a.*