HOW MANY PRIME NUMBERS ARE THERE?

(1) Let p_i denote the i-th prime number. So, for example, $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ Find p_i for $i = 4, 5, 6, 7$.
(2) Calculate the product $p_1 p_2 p_3$. Is this number prime? How do you know?
(3) Calculate $p_1 p_2 p_3 + 1$. Is this number prime? How do you know?
(4) In general, suppose you multiply the first k primes together and then add 1. (So in the problem above you multiplied the first three primes together and then added 1.) Will you always get a prime number this way? Why or why not?
(5) Find a positive number k such that $p_1 p_2 \cdots p_k + 1$ is *not* prime.
(6) Suppose that there only 100 prime numbers $p_1, p_2, \ldots, p_{100}$. Will $p_1 p_2 \cdots p_{100} + 1$ prime? Why or why not?
(7) Do you think there are an infinite or finite number of primes? Why?

When you are done with these questions, please turn over.

Theorem. There are an infinite number of primes.

The following statements constitute a proof of the theorem above, but they are out of order. Order them.

(1) Therefore, there are an infinite number of primes.
(2) Since $q > p_i$ for all i, q is not on our list of prime numbers.
(3) Let $p_1, p_2, p_3, \ldots, p_k$ be the list of primes where $p_1 < p_2 < \cdots < p_k$.
(4) Let k be the number of primes.
(5) This is a contradiction.
(6) Therefore, q is not prime.
(7) Suppose there are a finite number of primes,
(8) Therefore no p_i divides q.
(9) Let $q = p_1 p_2 \ldots p_k + 1$.
(10) Thus we have found a non-prime number that is not divisible by any prime.
(11) Therefore, our assumption that there are a finite number of primes is false.
(12) For all p_i, $q \bmod p_i = 1$ since if we divide q by p_i, we get a remainder of 1.