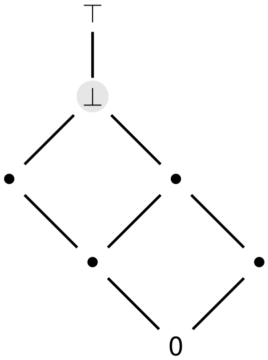
ICL: Intuitionistic Control Logic



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Outline

- Overview of Goals of ICL and its Basic Characteristics
- The Semantics of ICL: from Kripke Models to Categories
- The Interpretation of Proofs
- Sequent Calculus/Tableaux and Cut Elimination
- Natural Deduction and $\lambda\gamma$-calculus
- The Representations of *call/cc* and $\mathcal{C}$
- The Computational Content of Contraction and Disjunction
- Discussion of Related Systems.

Quick Summary of ICL:

- Propositional Logic with $\wedge$, $\vee$, $\supset$, $\top$, 0, and $\perp$.
- ***Identical to Intuitionistic Logic if $\perp$ removed***
- **Two forms of negation:** $\sim A = A \supset 0$; $\neg A = A \supset \perp$
- **Law of excluded middle:** $A \vee \neg A$ (but not $A \vee \sim A$)
- But **no involutive negation:** both $\neg\neg A \supset A$ and $\sim\sim A \supset A$ are unprovable. (but $\sim\neg A \supset A$ is provable).
- **No simple translation to linear logic:** not just $!A \multimap B$ plus $\perp$.
- Can also be described as intuitionistic logic plus a version of Peirce's law.
- Goals: good **semantics** and proof systems with **cut-reduction** procedures.

Kripke Semantics

All Models has a *Root* **r**: $\langle \mathbf{W}, \mathbf{r}, \preceq, \models \rangle$

- $u \models \top$; $u \not\models 0$
- $\mathbf{r} \not\models \bot$
- $q \models \bot$ **for all** $q \succ \mathbf{r}$
- $u \models A \wedge B$ iff $u \models A$ and $u \models B$
- $u \models A \vee B$ iff $u \models A$ or $u \models B$
- $u \models A \supset B$ iff for all $v \succeq u$, $v \not\models A$ or $v \models B$.

- A model $M \models A$ if and only if $\mathbf{r} \models A$ by monotonicity of $\models$.
- $\mathbf{r} \models A$ if and only if $\mathbf{r} \not\models \neg A$. Thus $\mathbf{r} \models A \vee \neg A$
- Neither 0 nor $\bot$ has a model (both inconsistent)

Other Important Properties of $\perp$

- **A formula that does not contain $\perp$ as a subformula is valid in ICL if and only if it is valid in intuitionistic logic.**

- Because $A \vee \neg A$ is valid, the disjunction property is guaranteed only for formulas that does not contain $\perp$.

- Formulas that contain $\perp$ can still have intuitionistic proofs ($\neg A \supset \neg A$).

- **No more need for *polarization***

The semantics of *disproofs*

What can this little model show?

$$\begin{tikzcd}
\mathbf{q} : \{B\} \\
\mathbf{r} : \{\;\} \arrow[u]
\end{tikzcd}$$

$\mathbf{r} \not\models A$; $\mathbf{r} \not\models B$; $\mathbf{q} \not\models A$, $\mathbf{q} \models B$

$\mathbf{r} \not\models {\sim} B \vee B$; $\mathbf{r} \not\models {\sim}{\sim} B \supset B$

$\mathbf{r} \not\models \neg\neg A \supset A$

(But note: $\mathbf{r} \models B \vee \neg B$ since $\mathbf{r}, \mathbf{q} \models \neg B$)

Sample Truths and Falsehoods

Valid	**Invalid**
$\neg A \vee A$	$\sim A \vee A$
$(\neg \mathbf{P} \supset \mathbf{P}) \supset \mathbf{P}$	$((P \supset Q) \supset P) \supset P$
$0 \supset A$	$\bot \supset A$
$\neg A \vee B \ \equiv\ \neg(A \wedge \neg B)$	$\sim A \vee B \ \equiv\ \sim(A \wedge \sim B)$
$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$	$\neg(A \wedge \neg B) \ \equiv\ \neg\neg(A \supset B)$
$\neg\neg A \ \equiv\ A \vee \bot$	$\sim\sim A \supset A$
$\sim\neg \mathbf{A} \supset \mathbf{A}$	$\neg\neg A \supset A$
$A \supset \neg \sim A$	$\neg \sim A \supset A$
$A \supset \neg\neg A$	$A \supset \sim\neg A$
$A \supset \sim\sim A$	$(\neg B \supset \neg A) \supset (A \supset B)$

Classical Logic inside ICL

- Define **Classical Implication $A \Rightarrow B$ as $\neg A \vee B$**
- $\neg A \vee B \equiv \neg(A \wedge \neg B)$, so **no “negative” translation needed**.
  ($\sim A \vee B$ does *not* represent classical implication.)
- **Hilbert’s axiom** $(\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$ holds.
- **A General Law of Admissible Rules:**
  if $A \Rightarrow B$ is valid, then A is valid implies B is also valid
- E.g., $\neg\neg A \supset A$ is invalid, but $\neg\neg A \Rightarrow A$ is valid, so $\dfrac{\neg\neg A}{A}$ is admissible
- ***Every classical implication corresponds to at least an admissible rule in ICL***

$\perp$ in Cartesian Closed Categories

- Let D be any **cartesian closed category** with products, coproducts, terminal object **T** and initial object **0**.
- Let **2** be the **two-element boolean algebra** represented as a category with two objects and three arrows: $\mathbf{2} : \mathit{false} \longrightarrow \mathit{true}$.
- Let $\mathbf{D_2}$ be a functor from D to **2**:
  - $\mathbf{D_2}(X) = \mathit{false}$ if X is uninhabited;
    $\mathbf{D_2}(X) = \mathit{true}$ if $\mathbf{T} \rightarrow X$ exists.
  - $\mathbf{D_2}(X \rightarrow Y) = \mathbf{D_2}(X) \rightarrow \mathbf{D_2}(y)$.
- Assume that D has a **right-adjoint $\mathbf{R_2}$ of $\mathbf{D_2}$:**
- Then $\mathbf{R_2}(\mathit{true}) \cong \mathbf{T}$ (terminal), as expected
- **but $\mathbf{R_2}(\mathit{false})$ is not isomorphic to 0 (initial).**

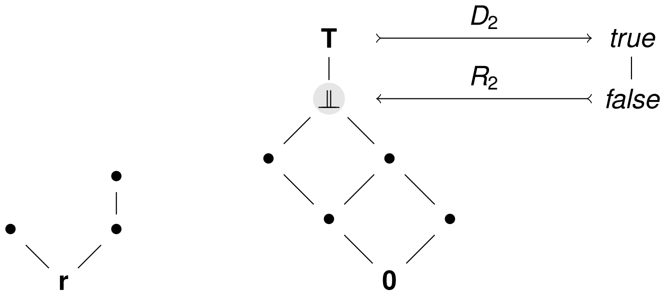
Let $\perp\!\!\!\perp$ = $\mathbf{R_2}(\mathit{false})$

- **$\perp\!\!\!\perp$ is a terminal object in the full subcategory of uninhabited objects of D.**
- **Essential Property of $\perp\!\!\!\perp$:**

  > ***For each object X in a category D with $\perp\!\!\!\perp$, X is uninhabited if and only if there is a unique arrow $\eta_X : X \to \perp\!\!\!\perp$.***

- Consequence: $\mathbf{T} \to A + \perp\!\!\!\perp^A$ **"does not not exist"**.
  ($A \vee \neg A$ is OK).
- "Constructive" semantics stops short here: more specific models required (in terms of Freyd covers). But this semantics can still be useful...
- There is no arrow from $\perp\!\!\!\perp^{(\perp\!\!\!\perp^A)}$ to A. No involutive negation.

From Categories to Kripke Models



Kripke Frame with Root, Heyting Algebra with **Second-Largest Point**, and Boolean Algebra **2**

How to Represent Proofs

λ **terms**	**Intuitionistic Logic**	**Classical Logic**
x	$\neg(A \vee \neg A)^d, A^x \vdash A$	$A \vdash \bot, A$
$\omega^l(x)$	$\overline{\neg(A \vee \neg A)^d, A^x \vdash A \vee \neg A}$	$\overline{\text{–MAGIC!–}}$
$[d]\omega^l(x)$	$\overline{\neg(A \vee \neg A)^d, A^x \vdash \bot}$	$\overline{A \vdash \bot,\ A \vee \neg A}$
$\lambda x.[d]\omega^l(x)$	$\overline{\neg(A \vee \neg A)^d \vdash \neg A}$	$\overline{\vdash \neg A,\ A \vee \neg A}$
$\omega^r(\lambda x.[d]\omega^l(x))$	$\overline{\neg(A \vee \neg A)^d \vdash A \vee \neg A}$	$\overline{\vdash A \vee \neg A,\ A \vee \neg A}$
$\lambda d.\omega^r(\lambda x.[d]\omega^l(x))$	$\overline{\vdash \neg(A \vee \neg A) \supset (A \vee \neg A)}$	$\overline{\vdash A \vee \neg A}$

$\lambda\gamma$ **term**	**Intuitionistic Control Logic:**
$\gamma(\lambda d.\omega^r(\lambda x.[d]\omega^l(x)))$	$\overline{\vdash A \vee \neg A}$

where $\gamma : (\neg P \supset P) \supset P$ *(our version of Peirce's Law)*

Categorically: $\gamma : P^{(\perp\!\!\!\perp^P)} \to P$ **“exists”** by the special property of $\perp\!\!\!\perp$.

Semantics of Proofs ***modulo*** γ

γ as a Natural Transformation (from François Lamarche)

- Define functor $\mathbf{F}(X) = X^{(\perp\!\!\!\perp^{X})}$.

  $\mathbf{F}(h : X \to Y) = \lambda K \lambda u.h(K(\lambda x.u(h(x)))) : X^{(\perp\!\!\!\perp^{X})} \to Y^{(\perp\!\!\!\perp^{Y})}$
- Then the collection of arrows γ is characterizable as a natural transformation from **F** to identity.

$$\begin{tikzcd}
\mathbf{F}(A) \arrow[r, "\gamma_A"] \arrow[d, "\mathbf{F}(h)"'] & \mathbf{I}_D(A) \arrow[d, "h"] \\
\mathbf{F}(B) \arrow[r, "\gamma_B"'] & \mathbf{I}_D(B)
\end{tikzcd}$$

$$h(\gamma_A) = \gamma_B(\mathbf{F}(h)) : A \to B.$$

How to represent γ as inference rule(s)

$$\frac{\neg B, \Gamma \vdash B}{\Gamma \vdash B}$$

or

$$\frac{\Gamma \vdash B;\ [B, \Delta]}{\Gamma \vdash B;\ [\Delta]}\ \mathit{Con} \qquad \frac{\Gamma \vdash B;\ [\Delta]}{\Gamma \vdash \bot;\ [B, \Delta]}\ \mathit{Esc}$$

- First version more accurate conceptually.
- Second set of rules enjoy better proof theoretic properties:
  1. preserves subformula property,
  2. clearly identifies intuitionistic/non-intuitionistic parts of proofs.
  3. clarifies cut reduction/normalization procedure.

γ in Proof Theory

$$\frac{\neg B, \Gamma \vdash B}{\Gamma \vdash B} \qquad \text{or as} \qquad \frac{\Gamma \vdash B;\ [B, \Delta]}{\Gamma \vdash B;\ [\Delta]}\ \mathit{Con} \qquad \text{and} \qquad \frac{\Gamma \vdash B;\ [\Delta]}{\Gamma \vdash \bot;\ [B, \Delta]}\ \mathit{Esc}$$

$$\dfrac{\dfrac{s :\ \neg B, \Gamma \vdash B}{\lambda d.s :\ \Gamma \vdash \neg B \supset B}\supset I \qquad \gamma :\ \vdash (\neg B \supset B) \supset B}{\gamma(\lambda d.s) :\ \Gamma \vdash B}\supset E\ (\textbf{cut})$$

- Write $\gamma(\lambda d.s)$ as just $\gamma d.s$
- $(\lambda x.s)\ t \longrightarrow_{\beta} s[t/x]$, but $(\gamma d.s)\ t \longrightarrow \gamma d.(s\{[d](w\ t)/[d]w\}\ t)$
- **Why is γ – or μ – still in the reduced term?**

- Because, in a sense, it represents a cut that cannot be eliminated, only permuted.

Sequent Calculus *LJC*

$$\frac{A, B, \Gamma \vdash C;\ [\Delta]}{A \wedge B, \Gamma \vdash C;\ [\Delta]}\ \wedge L \qquad \frac{A, \Gamma \vdash C;\ [\Delta] \quad B, \Gamma \vdash C;\ [\Delta]}{A \vee B, \Gamma \vdash C;\ [\Delta]}\ \vee L$$

$$\frac{\Gamma \vdash A;\ [\Delta] \quad B, \Gamma \vdash C;\ [\Delta]}{A \supset B, \Gamma \vdash C;\ [\Delta]}\ \supset L \qquad \frac{}{0, \Gamma \vdash A;\ [\Delta]}\ 0L \qquad \frac{}{\bot, \Gamma \vdash \bot;\ [\Delta]}\ \bot L$$

$$\frac{\Gamma \vdash A;\ [\Delta] \quad \Gamma \vdash B;\ [\Delta]}{\Gamma \vdash A \wedge B;\ [\Delta]}\ \wedge R \qquad \frac{\Gamma \vdash A;\ [\Delta]}{\Gamma \vdash A \vee B;\ [\Delta]}\ \vee R_1 \qquad \frac{\Gamma \vdash B;\ [\Delta]}{\Gamma \vdash A \vee B;\ [\Delta]}\ \vee R_2$$

$$\frac{A, \Gamma \vdash B;\ [\Delta]}{\Gamma \vdash A \supset B;\ [\Delta]}\ \supset R \qquad \frac{}{\Gamma \vdash \top;\ [\Delta]}\ \top R \qquad \frac{}{A, \Gamma \vdash A;\ [\Delta]}\ \mathit{Id}$$

$$\frac{\Gamma \vdash A;\ [A, \Delta]}{\Gamma \vdash A;\ [\Delta]}\ \mathit{Con} \qquad \frac{\Gamma \vdash A;\ [\Delta]}{\Gamma \vdash \bot;\ [A, \Delta]}\ \mathit{Esc}$$

Natural Deduction System *NJC* with terms: The $\supset$ Fragment

$$\frac{t : A^{x}, \Gamma \vdash B;\ [\Delta]}{(\lambda x.t) : \Gamma \vdash A \supset B;\ [\Delta]} \supset I \qquad \frac{t : \Gamma \vdash A \supset B;\ [\Delta] \quad s : \Gamma' \vdash A;\ [\Delta']}{(t\ s) : \Gamma\Gamma' \vdash B;\ [\Delta\Delta']} \supset E$$

$$\frac{s : \Gamma \vdash 0;\ [\Delta]}{abort\ s : \Gamma \vdash A;\ [\Delta]}\ 0E \qquad \frac{}{exit : \Gamma \vdash \top;\ [\Delta]} \top I \qquad \frac{}{x : A^{x}, \Gamma \vdash A;\ [\Delta]}\ Id$$

$$\frac{t : \Gamma \vdash A;\ [\Delta]}{[d]t : \Gamma \vdash \bot;\ [A^{d}, \Delta]}\ Esc \qquad \frac{u : \Gamma \vdash A;\ [A^{d}, \Delta]}{\gamma d.u : \Gamma \vdash A;\ [\Delta]}\ Con$$

The Con Escapes! (and $\bot$ is the key)

$$\frac{s : \Gamma \vdash B;\ [B^d, \Delta]}{\gamma d.s : \Gamma \vdash B;\ [\Delta]}\ \mathit{Con} \qquad\qquad \frac{r : \Gamma \vdash B;\ [\Delta]}{[d]r : \Gamma \vdash \bot;\ [B^d, \Delta]}\ \mathit{Esc}$$

- $B^d \in \Delta$ is possible in *Esc*. (Contraction inside Γ, Δ is free)
- **The rest of the rules are entirely intuitionistic (LJ or NJ)**
- If *Esc* not used, then proof is still intuitionistic (*Con* will be vacuous).
- Contrast *Con* with *Decide*/*Dereliction*/*Passivate* in classical proof systems:

$$\frac{\vdash \Gamma, P;\ P}{\vdash \Gamma, P;\ \_} \qquad \mathit{D} \text{ rule in LC}$$

Here the P *leaves* the stoup.

How to permute cut above *Con*?

$$\dfrac{\Gamma \vdash A;\ [\Delta] \quad A, \Gamma' \vdash B;\ [\Delta']}{\Gamma\Gamma' \vdash B;\ [\Delta\Delta']}\ \mathit{cut}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \vdash A;\ [A, \Delta_1]}{\Gamma_1 \vdash \bot;\ [A, \Delta_1]}\ \mathit{Esc}}{\vdots}}{\Gamma_2 \vdash A;\ [A, \Delta_2]}}{\Gamma_2 \vdash \bot;\ [A, \Delta_2]}\ \mathit{Esc}}{\vdots}}{\Gamma \vdash A;\ [A, \Delta]}}{\Gamma \vdash A;\ [\Delta]}\ \mathit{Con} \qquad A, \Gamma' \vdash B;\ [\Delta']}{\Gamma\Gamma' \vdash B;\ [\Delta\Delta']}\ \mathit{cut}$$

Problem: clashes with β-reduction (looses confluence)

To preserve confluence we can:

1. Adopt call-by-value reduction strategy.
2. **Require the contracted formula to be of the form $A \supset B$:**

$$\dfrac{\dfrac{s : \Gamma \vdash A \supset B;\ [(A \supset B)^d, \Delta]}{\gamma d.s : \Gamma \vdash A \supset B;\ [\Delta]}\ \mathit{Con} \qquad t : \Gamma' \vdash A;\ [\Delta']}{(\gamma d.s)\ t : \Gamma\Gamma' \vdash B;\ [\Delta\Delta']}\ \mathit{cut}$$

(Similar choice made in original $\lambda\mu$ calculus)

$$\dfrac{\dfrac{q : \ \Gamma_1 \vdash A \supset B;\ [\Delta_1] \qquad t : \Gamma' \vdash A;\ [\Delta']}{qt : \ \Gamma_1\Gamma' \vdash B;\ [\Delta_1\Delta']}\ cut}{[d](qt) : \ \Gamma_1\Gamma' \vdash \bot;\ [B^d, \Delta_1\Delta']}\ Esc$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\vdots}{r : \ \Gamma_2\Gamma' \vdash A \supset B;\ [B^d, \Delta_2\Delta']} \qquad t : \Gamma' \vdash A;\ [\Delta']}{rt : \ \Gamma_2\Gamma' \vdash B;\ [B^d, \Delta_2\Delta']}\ cut}{[d](rt) : \ \Gamma_2\Gamma' \vdash \bot;\ [B^d, \Delta_2\Delta']}\ Esc$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\vdots}{s\{[d](wt)/[d]w\} : \ \Gamma\Gamma' \vdash A \supset B;\ [B^d, \Delta\Delta']} \qquad t : \Gamma' \vdash A;\ [\Delta']}{(s\{[d](wt)/[d]w\}\ t) : \ \Gamma\Gamma' \vdash B;\ [B^d, \Delta\Delta']}\ cut}{\gamma d.(s\{[d](wt)/[d]w\}\ t) : \ \Gamma\Gamma' \vdash B;\ [\Delta\Delta']}\ Con$$

$$(\gamma d.s)\ t \ \longrightarrow\ \gamma d.(s\{[d](wt)/[d]w\}\ t)$$

$\lambda\gamma$ calculus

1. $(\lambda x.s)\ t \ \longrightarrow\ s[t/x]$. ($\beta$-reduction)
2. $(\gamma d.s)\ t \ \longrightarrow\ \gamma d.(s\{[d](w\ t)/[d]w\}\ t)$. ($\mu\gamma$-reduction)
3. $abort(s)\ t \ \longrightarrow\ abort(s)$. (aborted reduction)
4. $\gamma a.s \ \longrightarrow s$ when a does not appear free in s. (vacuous γ)
5. $\gamma a.\gamma b.s \ \longrightarrow\ \gamma a.s[a/b]$. ($\gamma$-renaming)
6. $[d]\gamma a.s \ \longrightarrow\ [d]s[d/a]$. ($\mu$-renaming)

Confluent and Strongly Normalizing

Renaming rules eliminate redundant contractions (*Cons*)

$$[d]\gamma a.s \longrightarrow [d]s[d/a] \quad \text{(also found in } \lambda\mu\text{):}$$

$$\dfrac{\dfrac{s : \Gamma \vdash A;\ [A^b, \Delta]}{\gamma b.s : \Gamma \vdash A;\ [\Delta]}\ \mathit{Con}}{[d]\gamma b.s : \Gamma \vdash \bot;\ [A^d, \Delta]}\ \mathit{Esc} \quad \longrightarrow \quad \dfrac{s[d/b] : \Gamma \vdash A;\ [A^d, \Delta]}{[d]s[d/b] : \Gamma \vdash \bot;\ [A^d, \Delta]}\ \mathit{Esc}$$

Because $A^d \in \Delta$ is possible (contraction inside $[\Delta]$ is always available).

$\gamma a.\gamma b.s \longrightarrow \gamma a.s[a/b]$**:** eliminates consecutive contractions.
$\gamma a.s \longrightarrow s$ when a is not free in s: **all intuitionistic proof terms reduce to λ-terms**

The Computational Content of Contraction: *call/cc* and $\mathcal{C}$ operators

Our version of Peirce's Law: $(\neg P \supset P) \supset P \;=\; ((P \supset \bot) \supset P) \supset P$**:**

$$
\dfrac{
  \dfrac{
    x : (\neg P \supset P)^x \vdash \neg P \supset P; [\,] \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{y : (\neg P \supset P)^x, P^y \vdash P;\ [\,]}\ \mathit{Id}
      }{[d]y : (\neg P \supset P)^x, P^y \vdash \bot;\ [P^d]}\ \mathit{Esc}
    }{\lambda y.[d]y : (\neg P \supset P)^x \vdash \neg P;\ [P^d]}\ \supset I
  }{
    \dfrac{(x\ \lambda y.[d]y) : (\neg P \supset P)^x \vdash P;\ [P^d]}{\gamma d.(x\ \lambda y.[d]y) : (\neg P \supset P)^x \vdash P;\ [\,]}\ \mathit{Con}
  }\ \supset E
}{\mathcal{K} = \lambda x.\gamma d.(x\ \lambda y.[d]y) :\ \vdash (\neg P \supset P) \supset P;\ [\,]}\ \supset I
$$

- $\mathcal{K} \;=\; \lambda x.\gamma(\lambda d.(x\ \lambda y.dy)) \;=_\eta\; \gamma$
- $(\mathcal{K}\ M\ k_1\ k_2) \;\longrightarrow\; \gamma d.(M\ \lambda y.[d](y\ k_1\ k_2))\ k_1\ k_2$
- Given $E[z] = (z\ k_1\ k_2)$**,** $E[\mathcal{K}M] \;\longrightarrow\; \gamma d.E[M(\lambda y.[d]E[y])]$

For the $\mathcal{C}$ operator, $\neg\neg A \supset A$ and $\sim\sim A \supset A$ are both unprovable.
But we have $\sim\neg A \supset A$ $= ((A \supset \bot) \supset 0) \supset A$:

$$\dfrac{\dfrac{\dfrac{x :\sim\neg A^x \vdash \sim\neg A; [\,] \qquad \dfrac{\dfrac{\overline{y :\sim\neg A^x, A^y \vdash A; [\,]}\;Id}{[d]y :\sim\neg A^x, A^y \vdash \bot;\ [A^d]}\;Esc}{\lambda y.[d]y :\sim\neg A^x \vdash \neg A;\ [A^d]}\;\supset I}{x\ \lambda y.[d]y :\sim\neg A^x \vdash 0;\ [A^d]}\;\supset E}{\dfrac{abort\ (x\ \lambda y.[d]y) :\sim\neg A^x \vdash A;\ [A^d]}{\gamma d.abort\ (x\ \lambda y.[d]y) :\sim\neg A^x \vdash A;\ [\,]}\;Con}\;0E}{\mathcal{C} = \lambda x.\gamma d.abort\ (x\ \lambda y.[d]y) : \vdash \sim\neg A \supset A;\ [\,]}\;\supset I$$

- $(\mathcal{C} M\ k_1\ k_2) \ = \ \gamma d.abort(M\ \lambda y.[d](y\ k_1\ k_2))$
- $\mathcal{C} M = \mathcal{K}(\lambda k.abort(Mk))$
- $\mathcal{C}(\lambda z.M) = abort(M)$ for z not free in M
- *abort* replaces free variable φ in $\lambda x.\mu\alpha.[\varphi](x\ \lambda y.\mu\delta.[\alpha]y)$ (original $\lambda\mu$ term).

NJC with Non-Additive Disjunction (partial future work)

$$\frac{s : \Gamma \vdash A;\ [\Delta] \quad t : \Gamma' \vdash B;\ [\Delta']}{(s, t) : \Gamma\Gamma' \vdash A \wedge B;\ [\Delta\Delta']}\ \wedge I \qquad \frac{s : \Gamma \vdash A \wedge B;\ [\Delta]}{(s)_\ell : \Gamma \vdash A;\ [\Delta]}\ \wedge E_1 \qquad \frac{s : \Gamma \vdash A \wedge B;\ [\Delta]}{(s)_r : \Gamma \vdash B;\ [\Delta]}\ \wedge E_2$$

$$\frac{s : \Gamma \vdash A;\ [B^d, \Delta]}{\omega^\ell d.s : \Gamma \vdash A \vee B;\ [\Delta]}\ \vee I_1 \qquad \frac{s : \Gamma \vdash B;\ [A^d, \Delta]}{\omega^r d.s : \Gamma \vdash A \vee B;\ [\Delta]}\ \vee I_2$$

$$\frac{v : \Gamma_1 \vdash A \vee B;\ [\Delta_1] \quad s : A^x, \Gamma_2 \vdash C;\ [\Delta_2] \quad t : B^y, \Gamma_3 \vdash C;\ [\Delta_3]}{(\lambda x.s, \lambda y.t)\ v : \Gamma_1\Gamma_2\Gamma_3 \vdash C;\ [\Delta_1\Delta_2\Delta_3]}\ \vee E$$

- $(u, v)\ (\omega^\ell d.t) \longrightarrow \gamma d.(u\ t\{[d](v\ w)/[d]w\});$
  $(u, v)\ (\omega^r d.t) \longrightarrow \gamma d.(v\ t\{[d](u\ w)/[d]w\})$ (ω-reduction)
- $(u, v)\ \gamma d.t \longrightarrow \gamma d.(u, v)\ t\{[d](u, v)w/[d]w\}$ ($\omega\gamma$-reduction)
- $(u, v)_\ell \longrightarrow u;\ (u, v)_r \longrightarrow v.$ (projections)
- $(\gamma d.s)_\ell \longrightarrow \gamma d.s_\ell\{[d]w_\ell/[d]w\};$
  $(\gamma d.s)_r \longrightarrow \gamma d.s_r\{[d]w_r/[d]w\}.$ (γ-projections)

$$\dfrac{\dfrac{u : \Gamma \vdash A;\ [B^d, \Delta]}{\omega^\ell d.u : \Gamma \vdash A \vee B;\ [\Delta]}\ \vee I_1 \qquad s : A^x, \Gamma \vdash C;\ [\Delta] \qquad t : B^y, \Gamma \vdash C;\ [\Delta]}{(\lambda x.s, \lambda y.t)\ \omega^\ell d.u : \Gamma \vdash C;\ [\Delta]}\ \vee E\ (\textbf{cut})$$

Reduces to:

$$\dfrac{\dfrac{\dfrac{u : \Gamma \vdash A;\ [B^d, \Delta] \qquad t : B^y, \Gamma \vdash C;\ [\Delta]}{u\{[d](\lambda y.t)w/[d]w\} : \Gamma \vdash A;\ [C^d, \Delta]}\ cut_2 \qquad s : A^x, \Gamma \vdash C;\ [\Delta]}{(\lambda x.s)\ u\{[d](\lambda y.t)w/[d]w\} : \Gamma \vdash C;\ [C^d, \Delta]}\ cut}{\gamma d.(\lambda x.s)\ u\{[d](\lambda y.t)w/[d]w\} : \Gamma \vdash C;\ [\Delta]}\ Con$$

$$(u, v)\ (\omega^\ell d.t) \ \longrightarrow\ \gamma d.(u\ t\{[d](v\ w)/[d]w\})$$

The Computational Content of Disjunction

public int f(String s) throws IOEXCEPTION

try $(\lambda z.t)s$ *catch exception e with* $\lambda y.u$*.*

$(\lambda x.(x\ s),\ \lambda y.u)\ (\omega^{\ell} d.\lambda z.t)$.

- x is not free in s: reverses application: $(\lambda x.x\ s)t = (t\ s)$
- $\omega^{\ell} d.\lambda z.t\ \ :\ \ (\mathbf{A} \supset \mathbf{C}) \vee \mathbf{B}$
- *Exception handler* $\lambda y.u\ \ :\ \ \mathbf{B} \supset \mathbf{C}$
- Term reduces to $\gamma d.t\{[d]\lambda(y.u)e/[d]e\}[s/z]\ \ :\ \ \mathbf{C}$
- $[d]e$ throws exception
- Reduces to $t[s/z]$ if no exceptions thrown (vacuous γ).
- **$\vee$-elimination replaces $\supset$-elimination for such procedures.**

Comparisons: Ong and Stewart's $\lambda\mu$

$$\frac{\Gamma; \Delta \vdash s : A}{\Gamma; \Delta, A^{\alpha} \vdash [\alpha^{A}]s : \bot}\ \bot - \mathit{intro} \qquad \frac{\Gamma; \Delta, B^{\beta} \vdash s : \bot}{\Gamma; \Delta \vdash \mu\beta^{B}.s : B}\ \bot - \mathit{elim}$$

- $\bot$-intro is very similar to *Esc*, but what is $\bot$-elim?
- "$\bot$" appears to be playing **two different roles:** enables contraction **and** 0-elimination ($0 \supset A$).
- The $\neg\neg A \Rightarrow A$ has fine proof (no free variables)
- **But why should Peirce's law require $\bot$-elim?**
- The computational content of Peirce's law is not attributed to contraction. $(\neg P \supset P) \supset P$ *is* contraction.

Comparisons: Girard's *LC*

Similarities:

- Formula must **stay in the stoup** until something *significant* happens.
- $\perp$ is **"negative"**; the other constants and atoms are **"positive"**

  $A \wedge B$ is negative if both A and B are negative, else positive.

  $A \vee B$ is negative if either A or B is negative, else positive.
- **negative** means *Esc* rule is possible; **positive** means purely intuitionistic.

Differences:

- LC does not contain **intuitionistic implication:**

  In ICL, $A \supset B$ is negative if B is negative, else positive.
- **Polarization not needed in ICL.** No built-in "dual" atoms $A^{\perp}$.
- Relationship to focusing (focalisation) also lost with $\supset$.

Can ICL be translated into linear logic?

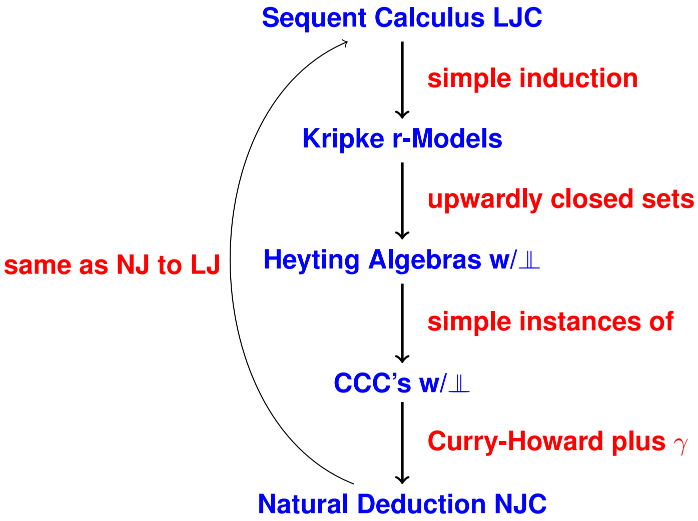
- Just translate IL into LL around the formula $!A \multimap B$, then “throw in” $\bot$. **Not even close!**
- $A \lor \neg A \stackrel{?}{=} A \oplus (!A \multimap \bot)$: linear formula not provable.
- Better attempt: use a **polarized** translation (like LC’s):

  Recognize $A \lor \neg A$ as **negative**, then use $A \parr (!A \multimap \bot)$.
- Still doesn’t work for Peirce’s formula: $(\neg P \supset P) \supset P$: **need contraction on P**

  Not as long as $\supset$ is translated using $!A \multimap B$.
- Only apparent solution: use **classical implication:** $(\neg P \Rightarrow P) \Rightarrow P$ where $A \Rightarrow B = !A \multimap ?B$. **But when to use $\Rightarrow$ instead of $\supset$?**
- **What can we conclude, if no reasonable translation exists?**

  Linear logic is not subtle enough to go ***in between*** *intuitionistic and classical logic.*

Soundness and Completeness



Sequent Calculus LJC

simple induction

Kripke r-Models

upwardly closed sets

Heyting Algebras w/$\perp\!\!\!\perp$

simple instances of

CCC's w/$\perp\!\!\!\perp$

Curry-Howard plus γ

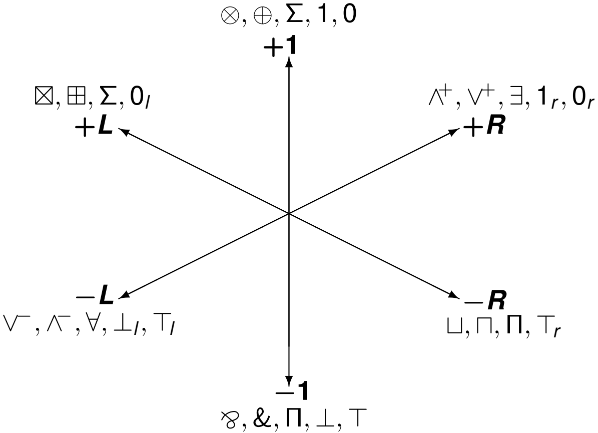
Natural Deduction NJC

same as NJ to LJ

Where did ICL came from:

Attempt to find a ***unified logic***

- Starting point: Girard's system **LU**.
- Our early attempt at an unified logic, **LUF**:



$\otimes, \oplus, \Sigma, 1, 0$

$\mathbf{+1}$

$\boxtimes, \boxplus, \Sigma, 0_l$

$+\boldsymbol{L}$

$\wedge^{+}, \vee^{+}, \exists, 1_r, 0_r$

$+\boldsymbol{R}$

$-\boldsymbol{L}$

$\vee^{-}, \wedge^{-}, \forall, \bot_l, \top_l$

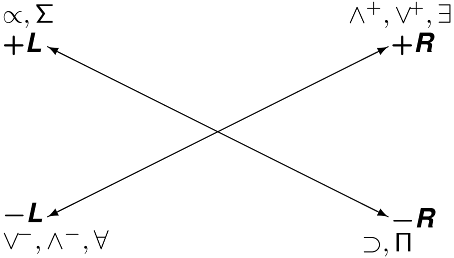
$-\boldsymbol{R}$

$\sqcup, \sqcap, \Pi, \top_r$

$\mathbf{-1}$

$⅋, \&, \Pi, \bot, \top$

- Second attempt at a unified logic: **PIL**:



- The proof theory of PIL contained both LJ and LC.
- **Breakthrough: found Kripke Semantics for PIL**
- Possible to unify classical and intuitionistic logics inside an intuitionistic semantics.
- **The identification of $\perp$ as a constant, which makes $A \vee \neg A$ possible, replaced the need for polarized connectives.**