Combining Classical Logic and Intuitionistic Logic

Double Negation, Polarization, Focusing, and Semantics

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Challenges of Combining Intuitionistic Logic with Classical Logic Using a Double Negation Translation

- Intuitionistic implication should not collapse into classical implication. Consider \( A \lor^e (B \supset C) \rightarrow \neg (\neg A \land \neg (B \supset C)) \)

- How to distinguish the introduction of a translated classical "connective" from intuitionistic introductions.

  Introduction of classical disjunction \( A \lor^e B \):

  \[
  \begin{align*}
  \neg A, \neg B, \Gamma \vdash \\
  \neg A \land \neg B, \Gamma \vdash \\
  \Gamma \vdash \neg (\neg A \land \neg B)
  \end{align*}
  \]

  No guarantee that sequence won’t be interrupted by other rules.

- How to recognize classical "dualities". How is \( \neg (\neg A \land \neg B) \) the "dual" of \( \neg (\neg A \land \neg B) \) in an intuitionistic sense, given that \( \neg \neg P \neq P \).
Challenges Continued ...

- How to distinguish classical from intuitionistic equivalence.
- Classical versus Intuitionistic Cut Elimination.

\[
\frac{A, \Gamma \vdash B \sim A, \Gamma \vdash B}{\Gamma \vdash B}
\]

Admissible in classical logic but not in intuitionistic logic.

*How do we simulate this cut in an intuitionistic proof system?*

- What is the *meaning* of mixed formulas such as \( A \vee^e (B \supset (C \vee D)) \)?
Outline and Overview:

- **Goal:** Combine LJ and LC into *Polarized Intuitionistic Logic*

- LC does not include intuitionistic implication

- Start with Intuitionistic Logic with a designated atom $\bot$.

  - $\bot$ is just minimal "false" - this logic predates ICL.
    - ICL is a stand-alone logic
    - PIL *combine* logics

- Assign labels, i.e., "polarities" to formulas.

- Define Double-Negation translation.

- Use focusing (focalization) to isolate "classical connectives"

- Derive Unified Sequent Calculus

- Define Kripke/Algebraic Semantics
Syntax and Colors

- Formulas freely generated from atoms, $\land$, $\lor$, $\supset$, $0$ and designated atom $\perp$.
- Define $\neg A = A \supset \perp$; $(A \supset 0 = \sim A)$
- Formulas are Red or Green as follows:
  - $A \land B$, $A \lor B$, $0$, and $A \supset B$ where $B \neq \perp$ are red.
  - All atoms are red, except $\perp$, which is green.
  - $\neg A$ ($A \supset \perp$) is green.
    - $\neg^{2n}(R)$, $n > 0$, are reddish green (also includes $\perp$)
    - $\neg^{2n+1}(R)$ are solidly green
- Red and Green formulas can be logically equivalent:
  - $(A \land B) \supset \perp \equiv A \supset (B \supset \perp)$
- This polarization is not same as duality in linear logic: $?X \dashv \vdash !Y$
Recovering Classical "Dualities"

- $M^\perp = \neg M$ for red or reddish-green $M$
- $(\neg M)^\perp = M$

**Syntactic Identity:** $A^{\perp\perp} = A$

- $A^\perp$ is convenient way to refer to doubly-negated formulas
- $A^\perp$ is not a connective.
- if $A \equiv B$, then $A^\perp$ is only *classically* $\equiv$ to $B^\perp$
  
  \[
  ((A \land B) \supset \perp)^\perp = A \land B, \quad (A \supset (B \supset \perp))^\perp = \neg (A \supset \neg B)
  \]
Double Negation as Macro Expansion

$R$ red and $E$ green

- $A \lor^e B = (A \perp \land B \perp) \perp = \neg(A \perp \land B \perp)$
- $A \land^e B = (A \perp \lor B \perp) \perp = \neg(A \perp \lor B \perp)$
- $1 = \perp \perp = \perp \triangledown \perp; \quad \top = 0 \perp = 0 \triangledown \perp$

To complete the definition of $A\perp$, we need missing link:

- $A \bowtie B = \neg(A \triangledown B \perp)$
  includes special case: $(R \bowtie 1) = \neg\neg R$

These are not yet new connectives, just labels
The following holds:

- $1^\perp = \bot$; $\top^\perp = 0$
- $(A \lor^e B)^\perp = A^\perp \land B^\perp$
- $(A \land^e B)^\perp = A^\perp \lor B^\perp$
- $(A \propto B)^\perp \equiv A \supset B^\perp \quad (\text{mod } \neg\neg\neg P \equiv \neg P)$

Caution: do not equate “green” with “classical.”

Classical fragment will use $\lor$ and $\lor^e$, $\land$ and $\land^e$, 0 and $\bot$, 1 and $\top$.

The classical fragment will be more LC than LK.
Intuitionistic Sequent Calculus LJ

\[
\begin{align*}
A, B, \Gamma \vdash D & \quad \wedge L \\
A \wedge B, \Gamma \vdash D & \quad \wedge R \\
A \top B, \Gamma \vdash D & \quad \top L \\
A \top B, \Gamma \vdash D & \quad \top R
\end{align*}
\]

\[
\begin{align*}
A, \Gamma \vdash D & \quad \vee L \\
A \vee B, \Gamma \vdash D & \quad \vee R
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A & \quad \wedge L \\
\Gamma \vdash B & \quad \wedge R \\
\Gamma \vdash A_1 \vee A_2 & \quad \vee L
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A & \quad \top L \\
\Gamma \vdash B & \quad \top R
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A_i & \quad \top R \\
\Gamma \vdash A_1 \vee A_2 & \quad \vee R
\end{align*}
\]

\[
\begin{align*}
a, \Gamma \vdash a & \quad \text{Id} \\
0, \Gamma \vdash D & \quad \text{0L} \\
\Gamma \vdash 1 & \quad \text{1R}
\end{align*}
\]

\[\bot \text{ is considered a special atom}\]
\(\lor^e, \land^e\) and \(\alpha\) as Synthetic Connectives in LJ?

**Distinguish between sequents** \(\Gamma \vdash A\) and \(\Gamma \vdash \bot\):  

Correspond to sequents with and without a *stoup*

**Introduction of a green formula** \(\neg A = A \supset \bot\):

\[
\frac{A, \Gamma \vdash \bot}{\Gamma \vdash A \supset \bot} \quad \quad \frac{A \supset \bot, \Gamma \vdash A}{\bot, \Gamma \vdash \bot} \quad \frac{\bot, \Gamma \vdash \bot}{\Gamma \vdash \bot} \quad \Gamma \vdash \bot \quad \frac{\bot, \Gamma \vdash B}{\Gamma \vdash B}
\]

But LJ is not good enough

We don’t want the following:

\[
\frac{B \supset C, \Gamma \vdash B}{B \supset C, \Gamma \vdash A \supset \bot} \quad \frac{A \supset \bot, \Gamma \vdash A}{\bot, \Gamma \vdash B}
\]
Looking for one-to-one mapping between proofs

Derive new introduction rules for $A \vee^e B = \neg(A \bot \land B \bot)$:

\[
\frac{\Gamma \vdash \bullet \, A, B}{\Gamma \vdash A \vee^e B} \quad \vee^e R \quad \frac{A \vee^e B, \Gamma \vdash \circ \, A \bot}{A \vee^e B, \Gamma \vdash \circ} \quad \vee^e L
\]

Want this to correspond one-to-one with the following fragments:

\[
\frac{A \bot, B \bot, \Gamma \vdash \bot}{A \bot \land B \bot, \Gamma \vdash \bot} \quad \frac{\neg(A \bot \land B \bot), \Gamma \vdash A \bot}{\neg(A \bot \land B \bot), \Gamma \vdash B \bot} \quad \frac{\neg(A \bot \land B \bot), \Gamma \vdash A \bot \land B \bot}{\bot, \Gamma \vdash \bot}
\]

Need focused intuitionistic sequent calculus (LJF)

even for *unfocused* synthetic introduction rules
A new dimension of polarization

- Atoms are “positive,” except \( \perp \), which is “negative”
- \( \lor, \land^+, 1 \) and 0 are positive
- \( \land^-, \supset \), are negative
- Positives are “synchronous” on the right; Negatives are synchronous on the left
- Asynchronous rules are always invertible
- Synchronous (and asynchronous) rules can be stringed together into a single phase.
- \[ A \lor^e B = (A^\perp \land^+ B^\perp)^\perp \]
- **Caution:** Do not confuse positive with red polarities: \( A \supset B \) is red but negative (red=positive only in LC)
Use Delays to Fine-Tune Focusing

\[ \partial^+(A) = A \wedge^+ 1; \quad \partial^-(A) = 1 \supset A \]

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<thead>
<tr>
<th>( F )</th>
<th>( F^\ell ) (left)</th>
<th>( F^r ) (right)</th>
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<td>( \partial^-(a) )</td>
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<td>( A \wedge B )</td>
<td>( \partial^+(A^\ell) \wedge^- \partial^+(B^\ell) )</td>
<td>( \partial^+(A^r \wedge^- B^r) )</td>
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<td>( A \supset B )</td>
<td>( \partial^-(A^r) \supset \partial^+(B^\ell) )</td>
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<td>( A \wedge^e B )</td>
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<tr>
<td>( A \lor^e B )</td>
<td>( \neg(\partial^-(A^\bot^r) \wedge^+ \partial^-(B^\bot^r)) )</td>
<td>( \neg\partial^-(A^\bot^\ell \wedge^+ B^\bot^\ell) )</td>
</tr>
<tr>
<td>( A \propto B )</td>
<td>( \neg(A^\ell \supset B^\bot^r) )</td>
<td>( \neg(\partial^-(A^r) \supset \partial^+(B^\bot^\ell)) )</td>
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Deriving the Sequent Calculus \( LP \)

**Different modes of sequents:**

- \( \Gamma \vdash \bullet \iff A_1^\perp, \ldots, A_n^\perp, \Gamma \vdash \bot \quad (\Gamma \vdash \bullet \iff \Gamma \vdash \bot) \)
- \( \Gamma \vdash \circ \iff \Gamma \vdash \circ \)

**Structural Rules** (\( R \) red, \( E \) green)

\[
\begin{align*}
\Gamma \vdash \bullet \hspace{1cm} & \quad \frac{\Gamma \vdash \bullet}{\Gamma \vdash \circ} \\
\Gamma \vdash \circ \hspace{1cm} & \quad \frac{\Gamma \vdash \bullet}{\Gamma \vdash \circ}
\end{align*}
\]

**Impredicative Rules**

\[
\begin{align*}
[\Gamma, \partial^-(A)] & \to [\bot] \\
[\Gamma, \partial^-(A)] & \to \bot \\
[\Gamma, \partial^-(A)] & \to \bot \\
[\Gamma] & \to \partial^-(A) \cup \bot
\end{align*}
\]

\[
\begin{align*}
[A \cup \bot, \Gamma] & \to A \\
[A \cup \bot, \Gamma] & \to [A \cup \bot, \Gamma] \vdash \bot \\
[A \cup \bot, \Gamma] & \to [A \cup \bot, \Gamma] \\
[A \cup \bot, \Gamma] & \to [\bot]
\end{align*}
\]
Correspondence between focusing phases and synthetic introduction rules must be relaxed:

\[ A \propto B \equiv (A \supset B^\perp) \supset \perp, \text{ which is } - \text{ followed by } + \]

\[-+ , +- \text{ are OK, but not } ++-+.\]
Sequent Calculus LP

Structural Rules and Identity

\[ \frac{\Gamma \vdash E}{\Gamma \vdash E} \quad \text{Signal} \]
\[ \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash A^\perp, \Theta} \quad \text{Store} \]
\[ \frac{A^\perp, \Gamma \vdash A}{\Gamma \vdash A} \quad \text{Load} \]
\[ \frac{a, \Gamma \vdash a}{\Gamma \vdash a} \quad \text{I} \]

Right-Red Introduction Rules

\[ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \quad \land R \]
\[ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \lor A_2} \quad \lor R \]
\[ \frac{A, \Gamma \vdash B}{\Gamma \vdash B} \quad \supset R \]

Left-Red Introduction Rules

\[ \frac{A, B, \Gamma \vdash R}{A \land B, \Gamma \vdash R} \quad \land L \]
\[ \frac{A, \Gamma \vdash R \quad B, \Gamma \vdash R}{A \lor B, \Gamma \vdash R} \quad \lor L \]
\[ \frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash R}{A \supset B, \Gamma \vdash R} \quad \supset L \]

Right-Green Introduction Rules

\[ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land^e B} \quad \land^e R \]
\[ \frac{\Gamma \vdash A, B}{\Gamma \vdash A \lor^e B} \quad \lor^e R \]
\[ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \propto B} \quad \propto R \]

Rules for Constants

\[ \frac{}{\Gamma \vdash 1} \quad 1R \]
\[ \frac{\Gamma \vdash R}{1, \Gamma \vdash R} \quad 1L \]
\[ \frac{0, \Gamma \vdash R}{0L} \]
\[ \frac{\Gamma \vdash \bot}{\bot R} \]
\[ \frac{\Gamma \vdash \top}{\top R} \]
Extends to First Order

Rules for Quantifiers

\[
\begin{align*}
\Gamma \vdash_0 A[t/x] & \quad \exists R \\
\Gamma \vdash_0 A & \quad \Pi R \\
A, \Gamma \vdash_0 R & \quad \exists L \\
\exists y. A, \Gamma \vdash_0 R & \quad \Pi L
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash_1 A[t/x] & \quad \Sigma R \\
\Gamma \vdash_1 \forall x. A & \quad \forall R
\end{align*}
\]

Here, \( y \) is not free in \( \Gamma \) and \( R \).

Why not remove delays and get focused LPF?

Possible, but first ...
LC Inside LP

\begin{align*}
\vdash \Gamma, N, P; S \\
\vdash \Gamma, N \lor P; S \\
\vdash \Gamma, N, P; \\
\vdash \Gamma, N \lor P; \\
\vdash \Gamma; P \vdash \Delta, N; \\
\vdash \Gamma \Delta; P \land N
\end{align*}

\begin{align*}
\Gamma, P, N \vdash S \\
\Gamma, P \land N \vdash S \\
\Gamma \vdash N, P \\
\Gamma \vdash N \lor^e P \\
\Gamma \vdash N \\
\Gamma \vdash P \\
\Gamma \vdash N \\
\Gamma \vdash P \land N
\end{align*}

\begin{align*}
\Gamma \vdash N \\
\Gamma \vdash N \\
\Gamma \vdash P \land N
\end{align*}

LC invariant: no “positive” introductions outside of the stoup

...subsumed by LP invariant: no green introduction in \( \vdash \) mode

LC is accidentally almost focused, but not LP
Independence from Double Negation Translation

- $\forall^e, \land^e, \not\!, \perp$ and $\top$ are now first-class green connectives and constants.

- $A^\perp$ is now De Morgan negation, defined by “dualities:”
  - $\forall^e/$ $\land$, $\land^e/$ $\lor$, $\not\!/$ $\supset$, $\perp/1$, $\top/0$,

- “dual atoms” $a/a^\perp$; Formulas are in negation normal form.
  - If $A \vdash_o B$ is provable, then $B^\perp \vdash_o A^\perp$ is provable.

- Reclassification of some formulas:
  - 1 and $R \supset \perp$ are now red. $(R \supset \perp)^\perp = R \not\! 1$

- Every green formula is of the form $R^\perp$ for some red $R$.
  - Given $A$ and $A^\perp$, one is red, the other is green.
Kripke Semantics

**Hybrid Model (Propositional Case):** \( \langle W, \preceq, C, \models \rangle \)

Requirements and definitions:

- \( \preceq \) is a transitive, reflexive ordering on non-empty set \( W \) of “possible worlds.”
- \( \models \) is a monotonic relation between elements of \( W \) and sets of atomic formulas.
- \( C \subseteq W \) (“classical worlds”)
- \( \Delta_u = \{ k \mid k \in C \text{ and } u \preceq k \} \) (”classical cover” of \( u \))
- required: \( \Delta_k = \{ k \} \) for all \( k \in C \). (for propositional models)
- if \( \Delta_u = \emptyset \) then \( u \) is imaginary.

*Every Kripke Model for IL is immediately a Hybrid Model, with a more structured interpretation of possible worlds.*
Rules of $\models$

for $u, v \in W; c, k \in C$, green $E$:

- $u \models 1$ and $u \not\models 0$
- $u \models A \lor B$ iff $u \models A$ or $u \models B$
- $u \models A \land B$ iff $u \models A$ and $u \models B$
- $u \models A \supset B$ iff for all $v \succeq u$, $v \models A$ implies $v \models B$
- $u \models E$ iff for all $k \in \triangle_u$, $k \models E$
- $c \models E$ iff $c \not\models E^\bot$

E.g., $c \models A \propto B$ iff $c \not\models A \supset B^\bot$ iff for some $v \succeq c$, $v \models A$ and $v \not\models B^\bot$.

Monotonicity preserved by condition $\triangle_c = \{c\}$.

• If $\triangle_u = \emptyset$, then $u \models E$ for all green $E$.
• $u \models A \lor^e A^\bot$
Important Countermodels

\[ s_1 : \{a, a^\perp\} \quad s_2 : \{a^\perp\} \]

\[ k : \{a^\perp\} \]

shows that \( a \lor^e \sim a \) and \( \sim a \lor^e \sim \sim a \) are not valid

shows that intuitionistic implication does not collapse

\[ k : \{p, q\} \]

\[ \uparrow \]

\[ s : \{\} \]

shows that \((p \land^e q) \supset p, (p \lor^e q) \supset (p \lor q)\), etc... are not valid:

\[
\frac{P \vdash R}{P \land^e Q \vdash R} \qquad \frac{P \vdash R \quad Q \vdash R}{P \lor^e Q \vdash R} \qquad \frac{P \land^e Q}{P} \]

\(\land L\) \quad \(\lor L\) \quad \(\land E\)

... are not valid inference rules; some device needed.
Semantics and Cut Admissibility

LP is sound/complete by Hintikka-Henkin constructions

Some admissible cuts guaranteed by semantics:

\[ \frac{\Gamma \vdash_\circ A \quad A, \Gamma' \vdash_\circ B}{\Gamma \Gamma' \vdash_\circ B} \quad \text{Cut} \quad \frac{A, \Gamma \vdash_\bullet \Theta \quad A^\perp, \Gamma' \vdash_\bullet \Theta'}{\Gamma \Gamma' \vdash_\bullet \Theta \Theta'} \quad \text{cut}_\bullet \quad \frac{\Gamma \vdash_\circ A \quad \Gamma' \vdash_\circ A^\perp}{\Gamma \Gamma' \vdash_\bullet} \quad \text{cut}_\perp \]

A non-admissible cut:

\[ \frac{\Gamma \vdash_\bullet P \quad P, \Gamma' \vdash_\circ Q}{\Gamma \Gamma' \vdash_\circ Q} \quad \text{bad cut} \]

when \( P, Q \) are red.

\[ k : \{ P, Q \} \]
\[ \uparrow \]
\[ s : \{ \} \]
Procedural Cut Elimination

\[
\begin{align*}
A^\perp, B^\perp, \Gamma \vdash & \quad \text{Store} \times 2 \\
\Gamma \vdash A, B & \quad \text{Load} \\
\Gamma \vdash A^e B & \quad \text{cut}
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A \lor B, \Gamma' \vdash A^\perp & \quad A \lor B, \Gamma' \vdash B^\perp \\
A \lor B, \Gamma' \vdash A^\perp \land B^\perp & \quad \land R \\
A \lor B, \Gamma' \vdash & \quad \text{cut} \\
\Gamma' \vdash A^\perp \land B^\perp & \quad \text{cut}
\end{align*}
\]

Reduces to:

\[
\begin{align*}
\Gamma \vdash A \lor B & \\
\Gamma \vdash A \lor B, \Gamma' \vdash B^\perp & \\
\Gamma' \vdash B^\perp & \quad \text{cut} \\
\Gamma' \vdash & \quad \text{cut}
\end{align*}
\]
Let’s Be Naive ...

\[ \frac{A, \Gamma \vdash R}{\Gamma \vdash A \land^e B, \Gamma \vdash R} \text{ naive-} \land^e L \]

Try to reduce the following cut:

\[
\begin{array}{c}
\Gamma \vdash A \\
\Gamma \vdash B
\end{array}
\frac{A \land^e B}{\Gamma \vdash A \land^e B} \text{ Signal}
\frac{A, \Gamma' \vdash R}{A \land^e B, \Gamma' \vdash R} \text{ naive-} \land^e L
\frac{\Gamma \vdash A}{\Gamma \vdash A \land^e B}
\frac{A \land^e B}{\Gamma \vdash A \land^e B} \text{ Cut}
\frac{\Gamma' \vdash R}{\Gamma' \vdash R}
\end{array}
\]

would require

\[
\begin{array}{c}
\Gamma \vdash A \\
A, \Gamma' \vdash R
\end{array}
\frac{A, \Gamma' \vdash R}{\Gamma' \vdash R} \text{ bad cut}
\]

Violates LP Invariant: no green introduction rules in \( \vdash \) mode
Alternative Proof System LPM

Right-Red Rules

\[
\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \lor B, \Delta} \quad \check{\lor} \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \land B, \Delta} \quad \check{\land} \quad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \supset B, \Delta} \quad \check{\supset} \quad \frac{\Gamma \vdash 1}{\Gamma \vdash 1, \Delta}
\]

Left-Red Rules

\[
\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \lor B, \Gamma \vdash \Delta} \quad \check{\lor} \quad \frac{A, B, \Gamma \vdash \Delta}{A \land B, \Gamma \vdash \Delta} \quad \check{\land} \quad \frac{A \supset B, \Gamma \vdash A \quad B, \Gamma \vdash \Delta}{\Gamma \vdash A \supset B, \Delta} \quad \check{\supset}
\]

Left-Green Rules

\[
\frac{A, \Gamma \vdash B, \Gamma \vdash \Delta}{A \lor^e B, \Gamma \vdash \Delta} \quad \check{\lor}^e \quad \frac{A, B, \Gamma \vdash \Delta}{A \land^e B, \Gamma \vdash \Delta} \quad \check{\land}^e \quad \frac{A, \Gamma \vdash B^\perp}{A \propto B, \Gamma \vdash \Delta} \quad \propto \quad \frac{\perp, \Gamma \vdash \perp}{\perp, \Gamma \vdash \perp}
\]

The Lift Rule and Identity

\[
\frac{E^\perp, \Gamma \vdash}{\Gamma \vdash E, \Delta} \quad \text{Lift} \quad \frac{a, \Gamma \vdash a, \Delta}{\text{l}_r} \quad \frac{a, a^\perp, \Gamma \vdash}{\text{l}_l} \quad \frac{0, \Gamma \vdash \Delta}{\text{0l}}
\]

\(E\) is a green formula and \(a\) is an atomic formula.
Some Properties of PIL

- $A \lor^e \neg A$ is valid/provable (LEM)
- if $A \lor B$ provable, either $A$ or $B$ provable (Disjunction Prop.)
- $A \alpha 1 \equiv A \lor^e \bot \equiv \neg \neg A$
- Classical and intuitionistic connectives can mix freely: In $A \lor^e (B \supset C)$, $\supset$ does not collapse

But limitations still exist...

- Define **classical implication**: $A \Rightarrow B = A \bot \lor^e B$:
  Can prove
  \[
  ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P
  \]
  \[
  ((P \Rightarrow Q) \supset P) \Rightarrow P
  \]
  \[
  ((P \supset \bot) \supset P) \Rightarrow P
  \]
  but not
  \[
  ((P \supset \bot) \supset P) \supset P
  \]
  And the outermost $\supset$ is most important.
Algebraic Perspective

\[ \perp = \{ u \in W : u \models \perp \} = \{ u \in W : \Delta u = \emptyset \} \]

Kripke Frame, \( C = \{ c, k \} \)  
Heyting Algebra with \( \perp \)  
Boolean Algebra \( 2^C \)

**Embedded Algebra**  
\[ = \{ K \cup \perp : K \subseteq C \} \]
Interpretation of formulas

- $h(1) = h(\top) = W; \quad h(\bot) = \bot$
- $h(A \lor B) = h(A) \sqcup h(B), \quad h(A \land B) = h(A) \sqcap h(B)$
- $h(A \supset B) = h(A) \to h(B)$.
- $h(R\bot) = h(R) \to \bot$ for all green $R\bot$.

Top of embedded boolean algebra = $C \cup \bot$.  (Alternatively, let $1 = C \cup \bot$, change $\Gamma \vdash_\circ 1$ to $\Gamma \vdash_\bullet 1$)

Define secondary interpretation $h'(A) = (h(\neg\neg A) \cap C) \cup \bot$:

- $h'(A \land B) = h'(A \land^e B) = h'(A) \cap h'(B)$
- $h'(A \lor B) = h'(A \lor^e B) = h'(A) \cup h'(B)$
- $\overline{h'(A)} = h'(A')$; $\overline{X}$ is boolean complement in embedded algebra.
- But $h'(A \supset B) \neq h'(A) \to h'(B)$
- $h'(E) = C \cup \bot$ iff $h(E) = T$ for green $E$. 
Black Hole ($\sim\sim$) versus Worm Hole ($\neg\neg$)

Black Hole: all points $A \to \emptyset$ (or $(A \to \emptyset) \to \emptyset$) (Glivenko 1929)

Not closed under $\lor$, closed under $\to$

$$(A \to \emptyset) \to (B \to \emptyset) \equiv ((A \to \emptyset) \land B) \to \emptyset.$$

No escape!

Worm Hole: all points $(A \cap C) \cup \bot$ (based on $\neg\neg A$)

Closed under $\lor$, not closed under $\to$
Semantic Alternatives; Conclusions

- Require $C \neq \emptyset$: so $\bot \neq T$
  But $\bot$ is no longer just an atom.

- Add red constant $\diamond = C \cup \bot$: $\diamond \approx ?1; \ k \models \diamond$ iff $k \in C$.

- Extend to first order quantifiers: lose property $\triangle_c = \{c\}$

- Let $\bot$ be the second-largest element: ICL
  $A \lor \neg A$ valid without a different version of disjunction.
Can a double-negation translation allow us to combine classical logic with intuitionistic logic?

Yes, polarize the doubly-negated formulas; then focus.

Derive sequent calculus LP with two modes \( \vdash \circ \) and \( \vdash \bullet \); satisfies cut-elimination.

Intuitionistic implication does not collapse in PIL.

\( A \equiv B \) intuitionistically if \( A \vdash \circ B \) and \( B \vdash \circ A \);
implies \( B \perp \vdash \bullet A \perp \) and \( A \perp \vdash \bullet B \perp \).

Semantics completes the lifting of labels into connectives; Defines new logic.

No need to involve linear logic.

\( ?A \perp \otimes B \) (\( !A \rightarrow \circ B \)) is properly linear (\( B \) is ”neutral”).

No neutrals needed; combination can occur within intuitionistic logic, with focusing and enriched semantics.
LPF: Focused LP

Separate positive/negative from red/green polarization

\[ \alpha, \Sigma, a^\perp \quad \land^+, \lor, \exists, 1, 0, a \]

+ Green

- Green

\[ \lor^e, \land^e, \lor, \bot, \top, a^\perp \]

- Red

\[ \lor, \land^-, \Pi, a^- \]
Can we *cross-focus* between +Green and +Red?

- + to +: OK (focusing in LJF).
- + to +: OK; \( A \not\propto (B \not\propto C) = \neg(A \supset (B \supset C^\perp)) \).
- + to +: OK; \( (A \lor B) \not\propto C = \neg((A \lor B) \supset C^\perp) \).
- + to +: **Not a chance!** \( A \lor (B \not\propto C) = A \lor \neg(B \supset C^\perp) \).

Pattern is + − +. In linear logic, \(!A\oplus\neg!(B \otimes C)\).

Need **two layers of focusing** with lateral transition rules.

\( \uparrow^\bullet/\downarrow^\bullet \) along -/+ axis.
\( \uparrow^\circ/\downarrow^\circ \) along -/+ axis.

\( \vdash_\circ \) corresponds to \( \uparrow^\circ, \downarrow^\bullet \).
\( \vdash_\bullet \) corresponds to \( \uparrow^\bullet, \downarrow^\circ \).
LPF (one sided version)

Structural/Reaction Rules

Lateral Reactions

\[
\frac{\Gamma : \Delta \uparrow^\bullet \Upsilon}{\Gamma : \Delta \uparrow^\circ \Upsilon} L\uparrow
\]

\[
\frac{\Gamma : \downarrow^\circ \Upsilon}{\Gamma : \downarrow^\bullet \Upsilon} L\downarrow
\]

Negative Reactions

\[
\frac{\Gamma : R \uparrow^\circ \Theta}{\Gamma : \uparrow^\circ R, \Theta} R_1\uparrow
\]

\[
\frac{\Gamma : \Delta \uparrow^\bullet D, \Theta}{\Gamma : \Delta \uparrow^\circ D, \Theta} R_2\uparrow
\]

\[
\frac{\Gamma : \downarrow^\circ S}{\Gamma : S \uparrow^n} D_1
\]

\[
\frac{\Gamma : \Delta \downarrow^n T}{\Gamma : \Delta \downarrow^\circ T} D_2
\]

Positive Reactions

\[
\frac{\Gamma : \Delta \uparrow^\bullet N}{\Gamma : \Delta \downarrow^\circ N} R_1\downarrow
\]

\[
\frac{\Gamma : \uparrow^\circ M}{\Gamma : \downarrow^\bullet M} R_2\downarrow
\]

\[
\frac{\Gamma : a \downarrow \downarrow^n a}{\Gamma : a \downarrow, \Gamma : \downarrow^n a} l_1
\]

\[
\frac{\Gamma : a \downarrow, \Gamma : \downarrow^n a}{\Gamma : a \downarrow, \Gamma : \downarrow^n a} l_2
\]

\[\Upsilon\] contains only green formulas; \(R\): red formula; \(D\): positive formula or negative green literal; \(S\): positive red formula; \(T\): positive formula; \(N\): negative green formula; \(M\): negative formula; \(a\), positive atom.
LPF Introduction Rules

**Constants**

\[
\begin{align*}
\vdash & \Gamma : \Delta \vdash \Theta \vdash & \\
\vdash & \Gamma : \Delta \vdash \bot, \Theta \vdash \bot & \\
\vdash & \Gamma : \Delta \vdash \top, \Theta \vdash \top & \\
\end{align*}
\]

**Negative Connectives**

\[
\begin{align*}
\vdash & \Gamma : \Delta \vdash A, B, \Theta \vdash A \lor B, \Theta \vdash & \\
\vdash & \Gamma : \Delta \vdash A, \Theta \vdash A \land B, \Theta \vdash & \\
\vdash & \Gamma : \Delta \vdash A, \Theta \vdash \forall x.A, \Theta \vdash & \\
\vdash & \Gamma : \Delta \vdash A, \gamma \vdash A, \gamma \vdash & \\
\vdash & \Gamma : \Delta \vdash B, \gamma \vdash B, \gamma \vdash & \\
\vdash & \Gamma : \Delta \vdash A \land B, \gamma \vdash A \land B, \gamma \vdash & \\
\vdash & \Gamma : \Delta \vdash B, \gamma \vdash B, \gamma \vdash & \\
\vdash & \Gamma : \Delta \vdash A \lor B, \gamma \vdash A \lor B, \gamma \vdash & \\
\vdash & \Gamma : \Delta \vdash A \lor B, \gamma \vdash A \lor B, \gamma \vdash & \\
\end{align*}
\]

\(x\) is not free in \(\Gamma, \Delta, \Theta; \gamma\) contains only green formulas

**Positive Connectives**

\[
\begin{align*}
\vdash & \Gamma : \downarrow A \vdash \Gamma : \downarrow B \vdash & \\
\vdash & \Gamma : \downarrow A_1 \vdash \Gamma : \downarrow A_2 \vdash & \\
\vdash & \Gamma : \downarrow A[t/y] \vdash & \\
\vdash & \Gamma : \downarrow D \vdash \Gamma : \downarrow B \vdash & \\
\vdash & \Gamma : \downarrow \Sigma y.A \vdash & \\
\vdash & \Gamma : \Delta \vdash \Delta \vdash \Sigma y.A \vdash & \\
\vdash & \Gamma : \Delta \vdash \Delta \vdash B \vdash & \\
\end{align*}
\]

\(\alpha \ (\subseteq L)\)