CSC/ENGG 185 Fall 2005, Assignment 5 Solutions
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Problem 1. (10pts) Page 84/2.

XmY - R.V. with joint pmf

$$f(x,y) = \frac{x+y}{30},\ x = 0,1,2,3; y = 0,1,2.$$

Not that this is a discrete distribution (defined over finite number points). We will use the pmf to find the probabilities of various events using

$$P((x,y) \in A) = \sum \sum_{(x,y)\in A} f(x,y),\ \text{where} A \subset \{(x,y)|x = 0,1,2,3, y = \ 0,1,2\}$$

a) $P(X \leq 2, Y = 1) = f(0,1) + f(1,1) + f(2,1) = 1/30 + 2/30 + 3/30 = 6/30 = 1/5$

b) $P(X > 2, Y \leq 1) = f(3,0) + f(3,1) = 3/30 + 4/30 = 7/30$

c)$P(X > Y) = f(1,0) + f(2,0) + f(2,1) + f(3,0) + f(3,1) + f(3,2) = (1 + 2 + 3 + 3 + 4 + 5)/30 = 18/30 = 3/5$

d)$P(X + Y = 4) = f(2,2) + f(3,1) = 4/30 + 4/30 = 4/15$

Problem 2. Page 84/6.

X, Y - R.V., life in years of two electric components. The joint density is

$$\begin{array}{rcll} f(x,y) & = & e^{-(x+y)}, & x > 0, y > 0 \\ f(x,y) & = & 0, & \text{otherwise} \end{array}$$

Find $P(0 < X < 1|Y = 2)$.

By definition, the conditional density $f_{X|Y}$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)},\ f_Y(y) > 0$$

and

$$P(0 < X < 1|Y = 1) = \int_0^1 f_{X|Y}(x,1)\,dx$$

For $y \geq 0$,

$$f_Y(y) = \int_0^\infty e^{-(x+y)}dx = e^{-y}\int_0^\infty e^{-x}dx = e^{-y}(-e^{-x})|_{x=0}^{\infty} = e^{-y}(-0+1) = e^{-y}$$

and

$$f_{X|Y}(x|y) = \frac{e^{-x-y}}{e^{-y}} = e^{-x},\ x \geq 0, y \geq 0$$

Thus

$$P(0 < X < 1|Y = 2) = \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1 = -e^{-1} + 1$$

Problem 3. Page 85/13

X - R.V., number malfunctions

Y - R.V., number times tech called in

The joint pmf $f(x,y)$ is

Y \ X	1	2	3	f_Y
1	0.05	0.05	0.1	0.2
2	0.05	0.1	0.35	0.5
3	0	0.2	0.1	0.3
f_X	0.1	0.35	0.55	

a),b) Find the marginal distributions.

The marginal of Y is obtained by summing the joint distribution over all X-values (for each row in the table above sum the columns), and the marginal of X is obtained by summing for each column the row values. The results are filled in the table (the most right column and the bottom row).

Quick check: the marginal pmf's values sum to 1.

c) Find $P(Y=3|X=2)$.

By definition of conditional distribution,

$$P(Y=3|X=2)=\frac{f(2,3)}{f_X(2)}$$

substituting the $f(2,3)$ and $f_X(2)$ from the table

$$P(Y=3|X=2)=0.2/0.35=20/35=4/7$$

Problem 4. Page 85/16.

A coin tossed 3 times.

X - R.V. number of heads

Y - R.V. number heads minor number tails

Find the joint distribution $f(x,y)$ od X and Y.

X takes values 0,1,2,3.

Y takes values -3,-1,1,3.

Since no specific information is given about the coin, we assume it is fair, so all outputs are equally likely with probability 1/8.

By definition,

$$f(x,y)=P(X=x,Y=y)$$

So,

$$
\begin{aligned}
f(0,-3) &= P(TTT \cap TTT) = P(TTT) = 1/8 \\
f(0,-1) &= P(TTT \cap \{TTH \cup THT \cup HTT\}) = P(\emptyset) = 0 \\
f(0,1) &= P(TTT \cap \{HHT \cup THH \cup HTH\}) = P(\emptyset) = 0 \\
f(0,3) &= P(TTT \cap HHH) = P(\emptyset) = 0 \\
f(1,-3) &= P(\{HTT, THT, TTH\} \cap TTT) = 0; \\
f(1,-1) &= P(\{HTT, THT, TTH\} \cap \{HTT, THT, TTH\}) = P(\{HTT, THT, TTH\}) = \\
&= P(HTT) + P(THT) + P(TTH) = 3/8 \\
f(1,1) &= P(\{HTT, THT, TTH\} \cap \{HHT, THH, HTH\}) = P(\emptyset) = 0 \\
f(1,3) &= P(\{HTT, THT, TTH\} \cap HHH) = 0 \\
f(2,-3) &= 0 \\
f(2,-1) &= 0 \\
f(2,1) &= P(\{HHT, THH, HTH\} \cap \{HHT, THH, HTH\}) = P(\{HHT, THH, HTH\}) = 3/8 \\
f(2,3) &= 0 \\
f(3,-3) &= 0 \\
f(3,-1) &= 0 \\
f(3,1) &= 0 \\
f(3,3) &= 1/8
\end{aligned}
$$

We can put $f(x, y)$ in a table

x \ y	-3	-1	1	3
0	1/8	0	0	0
1	0	3/8	0	0
2	0	0	3/8	0
3	0	0	0	1/8

Quick check: $\sum_{y=-3,-1,1,3} \sum_{x=0,1,2,3} f(x, y) = 1/8 + 3/8 + 3/8 + 1/8 = 1$.

Problem 5. Page 85/18.

A coin tossed twice.

Z - R.V., number of heads on first toss.

W - R.V., number heads on two tosses.

On one toss, H has 40% chance of occurring. Thus, on one toss $P(H) = 0.4, P(T) = 0.6$.

The sample space, $S = \{HH, HT, TH, TT\}$. We need to find the probability function$P$ on S. Since the outcomes of the two tosses are independent

$$
\begin{aligned}
P(HH) &= P(H)P(H) = 0.16 \\
P(HT) &= P(H)P(T) = 0.24 \\
P(TH) &= P(T)P(H) = 0.24 \\
P(TT) &= P(T)P(T) = 0.36
\end{aligned}
$$

Z takes the values $0, 1$.

W takes the values $0, 1, 2$.

a) Find the joint distribution $f(w, z)$ of W and Z.

By definition, $f(w, z) = P(W = w, Z = z)$.

$$
\begin{aligned}
f(0,0) &= P(W=0, Z=0) = P(TT \cap \{TH, TT\}) = P(TT) = 0.36 \\
f(0,1) &= P(W=0, Z=1) = P(TT \cap \{HT, HH\}) = 0 \\
f(1,0) &= P(W=1, Z=0) = P(\{HT, TH\} \cap \{TH, TT\}) = P(TH) = 0.24 \\
f(1,1) &= P(W=1, Z=1) = P(\{HT, TH\} \cap \{HT, HH\}) = P(HT) = 0.24 \\
f(2,0) &= P(W=2, Z=0) = P(HH \cap \{TH, TT\}) = P(\emptyset) = 0 \\
f(2,1) &= P(W=2, Z=1) = P(HH \cap \{HT, HH\}) = P(HH) = 0.16
\end{aligned}
$$

Quick Check: all values of $f(w, z)$ add up to 1.

Put in a table, $f(w, z)$ is

Z \ W	0	1	2	f_Z
0	0.36	0.24	0	0.6
1	0	0.24	0.16	0.4
f_W	0.36	0.48	0.16	

b) Find the marginal distribution of W, $f_W(w)$.

By definition,

$$f_W(w) = \sum_{z=0}^{1} f(w, z),$$

i.e., for fixed column (0,1, or 2), we have to sum all rows, the result f_W is given in the bottom row in the table above.

Quick check: $0.36 + 0.48 + 0.16 = 1$.

c) Find the marginal distribution of X, $F_Z(z)$.

By definition,

$$F_Z(z) = \sum_{w=0}^{2} f(w, z),$$

i.e., for fixed row ($z = 0, 1$), we sum all columns in the table above. The results are shown in the right most column f_Z.

Quick check, $0.4 + 0.6 = 1$.

d) Find $P(at\ least\ one\ head\ occurs)$,

$$P(at\ least\ one\ head\ occurs) = P(W \geq 1) = \sum_{w=1}^{2} f_W(w) = 0.48 + 0.16 = 0.64$$

Problem 6 Page 85/19. We did the problem in class. See your notes.

Problem 7 Page 85/20.

X and Y R.V. with a joint distribution given in the table for Problem 3. We have to answer the question if X and Y are independent or not.

If we fin x an y such that $f(x, y) \neq f_X(x) f_Y(y)$, then X and Y are not independent.

If for all x and y, $F(x, y) = f_X(x) f_Y(y)$, then X and Y are dependent.

Notice that $f(1, 3) = 0$, $f_X(1) = 0.1$, $f_Y(3) = 0.3$, so

$$0 = f(1, 3) \neq f_X(1) f_Y(3) = 0.02,$$

which means that X and Y are not independent.

Problem 8. Page 85/22.

The joint density of X and Y is

$$\begin{array}{rcll} f(x, y) & = & 6x, & 0 < x < 1, 0 < y < 1 - x \\ f(x, y) & = & 0, & \text{otherwise} \end{array}$$

a) Show that X an Y are not independent.

We have to check either if $f_{X|Y}(x|y) = f_X(x)$ for all x, or if $f(x, y) = f_X(x) f_Y(y)$, for all x and y.

We have to find the marginal distributions first, by integrating the joint density (we find f_X by integrating with respect to y, and f_Y by integrating it with respect to x).

$$\begin{array}{lcl} f_X(x) & = \int_{0<y<1-x} 6x\, dy,\ 0 < x < 1 & \\ f_X(x) & = & \int_0^{1-x} 6x\, dy, = \int_0^{1-x} 6x dy = 6xy|_0^{1-x} = 6x(1-x),\ 0 < x < 1 \\ f_X(x) & = & 0,\ \text{otherwise} \end{array}$$

Quick check:

$$\int_0^1 6x(1-x)\, dx = 3x^2 - 2x^3|_0^1 = 1$$

For the marginal density of Y,

$$\begin{aligned}
f_Y(y) &= \int_{0<x<1,0<y<1-x} 6x\,dx = \int_{0<x<1,0<x<1-y} 6x\,dx, 0<y<1 \\
&= 3x^2|_{x=0}^{1-y} = 3(1-y)^2 = 3(1-2y+y^2),\ 0<y<1 \\
f_Y(y) &= 0, \text{otherwise}
\end{aligned}$$

Quick check:

$$3\int_0^1 (1-2y+y^2)\,dy = 3(y-y^2+(1/3)y^3)\Big|_0^1 = 3(1-1+1/3) = 1$$

Now we can check if $f(x,y) = f_X(x)f_Y(y)$ for all $(x,y$. For $0<x<1, 0<y<1, x+y<1$,

$$f(x,y) = 6x;\ f_X(x) = 6x(1-x);\ f_Y(y) = 3(1-y)^2$$

and since, for example, for $x = 1/3, y = 1/3$,

$$\begin{aligned}
f(1/3,1/3) &= 6(1/3) = 2 \\
f_X(1/3)f_Y(1/3) &= 6(1/3)(1-1/3) \times 6(1-1/3)^2 = 4/3 \times 8/3 = 32/9
\end{aligned}$$

and thus

$$f(1/3,1/3) \neq f_X(1/3)f_Y(1/3)$$

Another way to check is to compute any of the conditional densities and show that they are functions of both x and y. And, yet another way you could get the same result is to observe that $f_{X|Y}(x|y)$ is not free from influence by y, since when the value of y is in $(0,1)$, the values that X can take depend on y, i.e. X is restricted to take values in $(0,1-y)$; thus knowing the value of Y we gain information about X. Thus X and Y are not independent. But beware, it is often easy to make mistakes when "reasoning" this way in general, checking the numbers ($f(x,y) = f_X(x)f_Y(y)$) or explicitly finding the expressions for the conditional densities always work. Also since we have to work on part b), we have to find the marginal and conditional distributions anyway.

b) Find $P(X > 0.3|Y = 0.5)$.

The conditional distribution is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

We have to be careful with the range of the integral here. The domain over which the conditional density is not 0 is specified by $0<x<1, 0<y<1, x+y<1$, for given y value (notice that this is the same as $0<x<1, 0<y<1-x$), so once $Y = 0.5$, $0 < x < 1-0.5 = 0.5$.

$$P(X > 0.3|Y = 0.5) = \int_{0.3}^{0.5} 6x/(3(1-0.5)^2)\,dx = 4x^2|_{0.3}^{0.5} = 1-0.36 = 0.64$$

Problem 9. Page 86/1. X - R.V., proportion Turkish, Y - R.V., proportion domestic; with joint density $f(x,y)$

$$\begin{aligned} f(x,y) &= 24xy, \quad 0 \le x \le 1, 0 \le y \le 1, x+y \le 1 \\ f(x,y) &= 0, \qquad \text{otherwise} \end{aligned}$$

a) Find $P(X > 0.5)$. Here, we need to find first the marginal $f_X(x)$ and then integrate it over $(0.5, 1]$.

By definition,

$$\begin{aligned} f_X(x) &= \int_{0 \le y \le 1, x+y \le 1} 24xy\, dy,\ 0 \le x \le 1 \\ f_X(x) &= \int_{0 \le y \le 1, y \le 1-x} 24xy\, dy = \int_{y=0}^{1-x} 24xy\, dy \\ &= 12xy^2|_{y=0}^{1-x} = 12x(1-x)^2, 0 \le x \le 1 \\ f_X(x) &= 0,\ \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} P(X > 0.5) &= \textstyle\int_{0.5}^1 12x(1-x)^2\, dx = \int_{0.5}^1 (12x - 24x^2 + 12x^3)\, dx = \\ &= (6x^2 - 8x^3 + 3x^4)|_{0.5}^1 = 1 - 6 \times 0.25 + 8 \times 0.125 - 3 \times 0.0625 = \\ &= 1 - 1.5 + 1 - 0.1875 = 0.3125 \end{aligned}$$

b) Find $f_Y(y)$.

By definition,

$$\begin{aligned} f_Y(y) &= \int_{0 \le x \le 1, x+y \le 1} 24xy\, dx = \int_0^{1-y} 24xy\, dx,\ 0 \le y \le 1 \\ f_Y(y) &= 12x^2y|_{x=0}^{1-y} = 12y(1-y)^2,\ 0 \le y \le 1 \\ f_Y(y) &= 0, \text{otherwise} \end{aligned}$$

Quick check: $\int_0^1 12y(1-y)^2 = 1$.

c) Find $P(X < 1/8 | Y = 3/4)$. First we need to find the conditional density $f_{X|Y}(x,y)$, and then integrate $f_{X|Y}(x, 3/4)$ over the interval $[0, 1/8)$.

By definition,

$$f_{X|Y}(x,y) = \frac{f(x,y)}{f_Y(y)},\ f_Y(y) > 0$$

Thus

$$\begin{aligned} f_{X|Y}(x,y) &= \tfrac{24xy}{12y(1-y)^2} = \tfrac{2x}{(1-y)^2}, && 0 \le x \le 1, x+y \le 1 \\ f_{X|Y}(x,3/4) &= \tfrac{2x}{(1-3/4)^2)}, && 0 \le x \le 1, x \le 1 - 3/4 \\ f_{X|Y}(x,3/4) &= \tfrac{2x}{1/16}, && 0 \le x \le 1/4 \\ f_{X|Y}(x,3/4) &= 32x, && 0 \le x \le 1/4 \\ f_{X|Y}(x,3/4) &= 0, && \text{otherwise} \end{aligned}$$

Quick check: $\int_0^{1/4} 32x\, x = 1$.

And finally,

$$P(X < 1/8 | Y = 3/4) = \int_0^{1/8} 32x\, dx = 16x^2|_0^{1/8} = 1/4$$

Problem 10. X R,V., number months between payments, with CDF

$$\begin{array}{rcll} F(x) & = & 0 & x < 1 \\ F(x) & = & 0.4 & 1 \le x < 3 \\ F(x) & = & 0.6 & 3 \le x < 5 \\ F(x) & = & 0.8 & 5 \le x < 7 \\ F(x) & = & 1 & x \ge 7 \end{array}$$

a) Find the probability distribution f of X. Note that F is a step function (with a graph like a stair case), it is not continuous. X is discrete R.V. The steps are of equal hight 0.2. At the points that F jumps, the pmf f will have non zero probability mass equal to the height of the step. Thus

$$\begin{array}{rcll} f(x) & = & 0.4 & x = 1 \\ f(x) & = & 0.2 & x = 3 \\ f(x) & = & 0.2 & x = 5 \\ f(x) & = & 0.2 & x = 7 \\ f(x) & = & 0 & otherwise \end{array}$$

You could get the same result without relying on the geometric intuition. We can pick any point at each $F(x) = 0$ to represent the "ground level", say $x_0 = 0$. First sort all the point at which F jumps and x_0

$$x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 5, x_4 = 7$$

Then by definition for $i = 1, 2, 3, 4$

$$f_(x_i) = F(x_i) - F(x_(i - 1)$$

and we obtain exactly the same resut as before.

b) We use, $P(a < X \le b) = F(b) - F(a), a < b$,

$$P(4 < X \le 7) = F(7) - F(4) = 1 - 0.6 = 0.4$$