CSC/ENGG 185 Fall 2005, Assignment 4 Solutions
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Problem 1. (10pts) Page 72/3.

Experiment: a coin tossed three times.

W – R.V. denoting number heads (H) minus number tails (T). The sample space is

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$$

s	TTT	TTH	THT	THH	HTT	HTH	HHT	HHH
W	-3	-1	-1	1	-1	1	1	3

Problem 2. Page 73/8. In one toss, head is twice as likely as head, so in one toss

$$P(H) = 2/3,\ P(T) = 1/3.$$

For the three-toss experiment, since the outcome in one toss is independent from the outcomes in the others

$$P(TTT) = P(T)P(T)P(T) = 1/27$$

$$P(TTH) = P(T)P(T)P(H) = 2/27$$

Similarly we compute the rest, the probability function on S is given below

s	TTT	TTH	THT	THH	HTT	HTH	HHT	HHH
P	1/27	2/27	2/27	4/27	2/27	4/27	4/27	8/27

A quick check: $P(s) \geq 0$ and $\sum_{s \in S} P(s) = 1$ – thus well defined probability function. Now that we have (S, P) we are ready to look for the pmf, f of W.

$$f(x) = P(X = x),\ x = -3, -1, 1, 3.$$

Then

$$\begin{aligned}
f(-3) &= P(X = -3) = P(TTT) = 1/27 \\
f(-1) &= P(X = -1) = P(TTH, THT, HTT) = P(TTH) + P(THT) + P(HTT) = 6/27 \\
f(1) &= P(X = 1) = P(THH, HTH, HHT) = P(THH) + P(HTH) + P(HHT) = 12/27 \\
f(3) &= P(X = 3) = P(HHH) = 8/27
\end{aligned}$$

Quick check: $\sum_{x=-3,-1,1,3} f(x) = 1/27 + 6/27 + 12/27 + 8/27 = 1$.

Problem 3. Page 73/11. 7 TV sets, 2 defective.

3 TV sets purchased.

X – R.V., represents the number of defective TV sets purchased. Find the pmf f and plot it.

X can take values $0, 1, 2$ (can't be 3 since the total number of defective among the 7 is 2).

$S = \{any 3 TV sets out of 7\}$.

Let C_k^n denotes the number of combinations when k items are chosen out of n regardless of the ordering.

$$C_k^n = \frac{n!}{k!(n-k)!},\ k = 0, 1, \ldots, n,\ n \text{ is any positive integer.}$$

Then

$$|S| = C_3^7 = 35,$$

i.e. the number of ways 3 TVs could be chosen out of 7 is 35. Since all outcomes are equally likely, $P(s) = 1/35$, for any $s \in S$.

Now that we have (S, P), we are ready to find the pmf f of X,

$$f(x) = P(X = x) = \frac{\#\text{ ways 3 TVs chosen with } x \text{ defective}}{35},\ x = 0, 1, 2;\ x \leq 2$$

The number ways the set of 3 could be chosen so that it contains exactly x defectives is $C_x^2.C_{(3-x)}^5$, $x \leq 2$

$$P(X = x) = \frac{C_x^2, C_{(3-x)}^5}{C_3^7} = \frac{2!5!}{x!(2-x)!(3-x)!(5-3+x)!}$$

Thus since the total number of ways 3 TVs could be chosen out of 7 is 35,

$$f(x) = P(X = x) = \frac{\frac{2!5!}{x!(2-x)!(3-x)!(5-3+x)!}}{35},\ x \leq 2.$$

The pmf f is given in the table

x	0	1	2
f	10/35	20/35	5/35

$f(x) = 0$ for $x \neq 0, 1, 2$, i.e.

$$\begin{aligned} f(x) &= 10/35, & x &= 0 \\ f(x) &= 20/35, & x &= 1 \\ f(x) &= 10/35, & x &= 2 \\ f(x) &= 0, & &\text{otherwise.} \end{aligned}$$

Quick check: $f(x) \geq 0$, $\sum_{x=0,1,2} f(x) = 10/35 + 20/35 + 5/35 = 1$.

The pmf f is graphed on Figure 1.

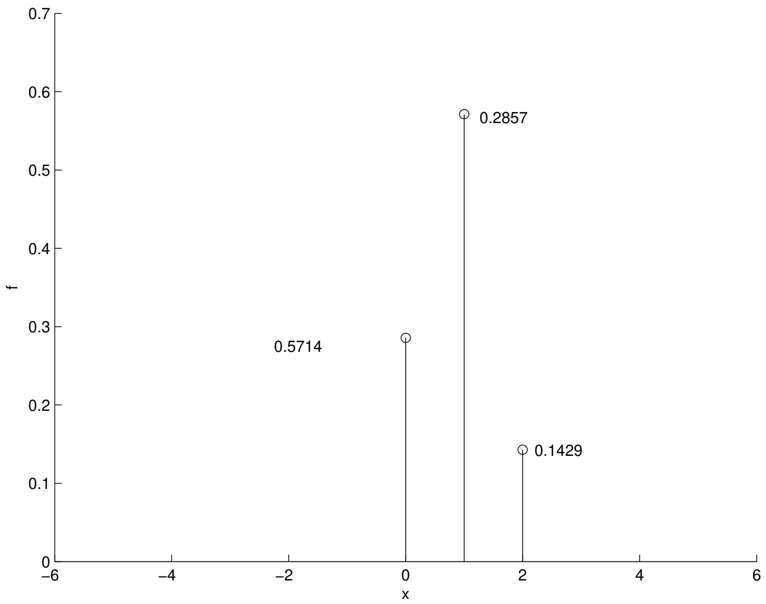

Figure 1: Graph of the pmf

Problem 4. Page 73/15.

Find the CDF F for the R.V. X from Problem 3.

By definition

$$F(x) = P(X \leq x)$$

and for a discrete random variable X

$$F(x) = \sum_{t \leq x} f(t)$$

Using the values in the table for $f(x)$ we obtain

x	0	1	2
F	10/35	30/35	1

$F(x) = 0, x < 0;\ F(x) = 1, x > 2.$, i.e.

$$\begin{aligned} F(x) &= 0, & x < 0 \\ F(x) &= 10/35, & 0 \leq x < 1 \\ F(x) &= 30/35, & 1 \leq x < 2 \\ F(x) &= 1, & x \geq 2. \end{aligned}$$

Quick check: $0 \leq F(x) \leq 1$, F is nondecreasing, $\lim_{x \to -\infty} = 0$, $\lim_{x \to \infty} = 1$.

The equations that connect probabilities to the CDF are

$$P(X \leq x) \quad = \quad F(x)$$

$$\begin{aligned} P(a < X \leq b) &= F(b) - F(a),\ a < b \\ P(X > x) &= 1 - F(x) \\ P(X = x_i) &= F(x_i) - F(x_{i-1}),\ \text{where}\{x_i\} \text{ are the sorted values at which } F \text{ jumps} \end{aligned}$$

a)

$$P(X = 1) = F(1) - F(0) = 30/35 - 10/35 = 20/35$$

b)

$$P(0 < X \leq 2) = F(2) - F(0) = 1 - 10/35 = 25/35;$$

Problem 5. Page 73/16. The graph of the CDF F from Problem 4 is given in Figure 2

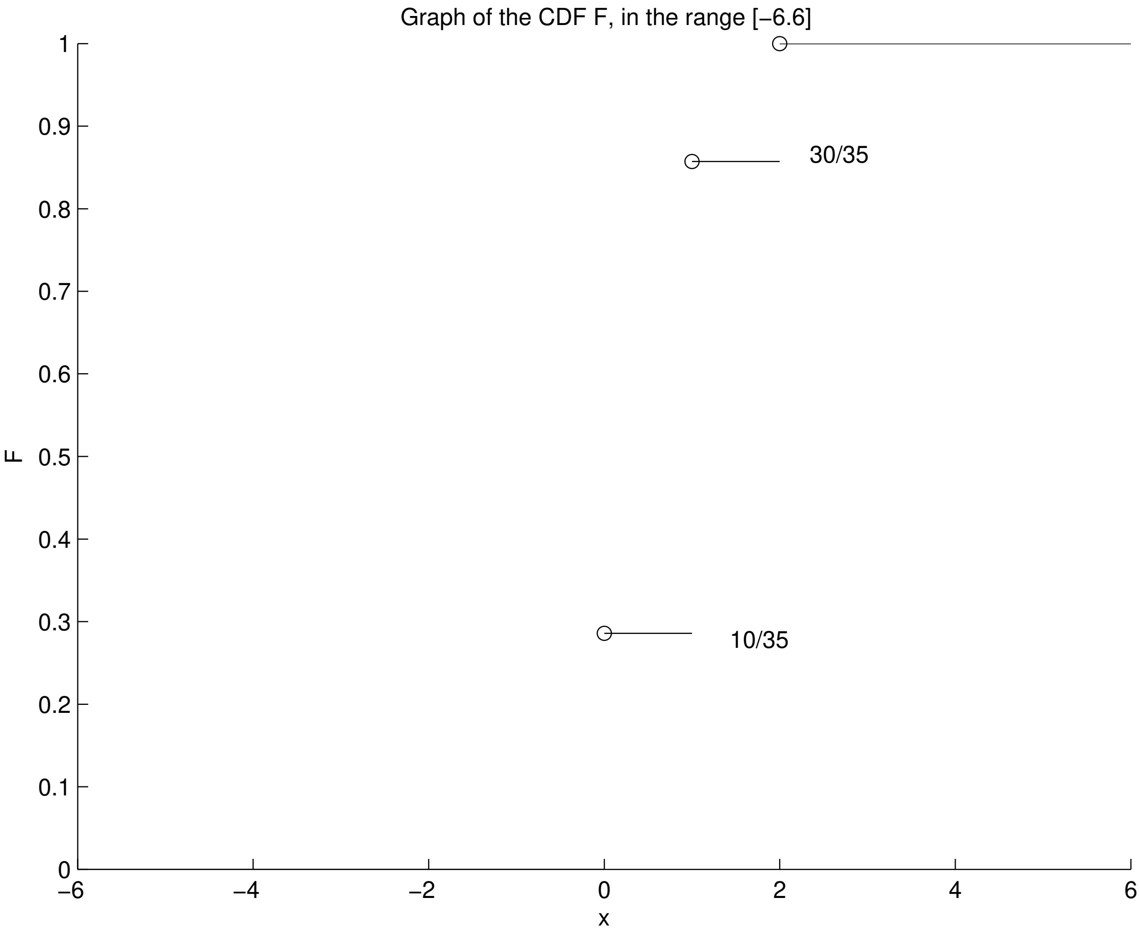

Figure 2: Graph of the CDF

Problem 6. Page 73/12. In the solutions below I used the name X instead of T for the random variable.

$$\begin{array}{rcll} F(x) & = & 0, & x < 0 \\ F(x) & = & 1/4, & 1 \le x < 3 \\ F(x) & = & 1/2, & 3 \le x < 5 \\ F(x) & = & 3/4, & 5 \le x < 7 \\ Fx) & = & 1, & x \ge 7. \end{array}$$

Note that F is a step function (the graph looks like staircase), thus X is a discrete random variable. The equations that connect probabilities to the CDF are

$$\begin{array}{rcl} P(X \le x) & = & F(x) \\ P(a < X \le b) & = & F(b) - F(a),\ a < b \\ P(X > x) & = & 1 - F(x) \\ P(X = x_i) & = & F(x_i) - F(x_{i-1}),\ \text{where}\{x_i\} \text{ are the sorted values at which } F \text{ jumps} \end{array}$$

a)

$$P(X = 5) = F(5) - F(3) = 3/4 - 1/2 = 1/4$$

b)

$$P(T > 3) = 1 - F(3) = 1 - 1/2 = 1/2$$

c)

$$P(1.4 < X < 6) = F(6) - F(1.4) = 3/4 - 1/4 = 1/2$$

Problem 7. Page 73/7. x is in units 100 hours; X is R.V. has the density

$$\begin{array}{rcll} f(x) & = & x, & 0 < x < 1 \\ f(x) & = & 2 - x, & 1 \le x < 3 \\ f(x) & = & 0, & \text{otherwise} \end{array}$$

The expression that connects the density to probabilities is:

$$P(a < X < b) = \int_a^b f(x)\,dx,\ a < b$$

and more generally,

$$P(x \in A) = \int_{x \in A} f(x)\,dx$$

Note that $\le$ and $<$ could be interchanged since X is continuous R.V.

a) Probability that the family runs the vacuum cleaner less than 120 hours. Since x is in units 100 hours we look for

$$\begin{array}{rcl} P(X \le 1.2) & = & \displaystyle\int_{-\infty}^{1.2} f(x)\,dx = \int_0^1 x\,dx + \int_1^{1.2} (2 - x)\,dx = \\ & = & \displaystyle\frac{x^2}{2}\Big|_{x=0}^{x=1} + (2x - \frac{x^2}{2})\Big|_{x=1}^{x=1.2} = \\ & = & 1/2 + 2.4 - 2 - 1.44/2 + 1/2 = 1 + 0.4 - 0.72 = 0.68 \end{array}$$

b) Since x is in units 100 hours, we look for

$$\begin{aligned} P(0.5 < X < 1) &= \int_{0.5}^{1} x\,dx = &(1)\\ &= 0.5x^2|_{x=0.5}^{x=1} &(2)\\ &= 0.5 - 0.125 = &(3)\\ &= 0.375 &(4) \end{aligned}$$

Problem 8. Page 74/17. X is continuous R.V., $f(x) = 1/2, x < 1 < 3$

a)

$$\int_1^3 \frac{1}{2} dx = \left.\frac{x}{2}\right|_{x=1}^{x=3} = 3/2 - 1/2 = 1$$

b)

$$P(2 < X < 2.5) = \int_2^{2.5} 0.5\,dx = 0.5x\Big|_{x=2}^{x=2.5} = 0.5(2.5-2) = 0.25$$

c)

$$P(X \leq 1.6) = \int_{-\infty}^{1.6} 0.5\,dx = \int_1^{1.6} 0.5\,x = 0.5x\Big|_{x=1}^{x=1.6} = 0.5(0.6) = 0.3$$

Problem 9. Page 61/1.

90% of guilty properly judged

10% of guilty improperly judged

1% innocent misjudged

One person selected at random from a groups in which only 5% have committed a crime.

Find $P(innocent|judged\ guilty)$.

Note that we know the probabilities of judging guilty or not given that we know if the person has committed a crime or not. What we are looking for is the reverse conditioning, i.e. the probability of being innocent, given that judged as guilty. This the first indication to use the Bayes rule. S consists of all people in the group, and it is partitioned into two events $S = B_1 \cup B_2$ where B_1 consists of all people that did not commit a crime, and B_2 of all who did. Thus we have to use the full Bayes Theorem for a sample space that is partitioned. Let A be the event that the person is judged guilty. We are looking for $P(B_1|A)$.

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

and

$$\begin{aligned} P(B_1|A) &= \frac{P(B_1)P(A|B_1)}{P(A)} = &(5)\\ &= \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)} = &(6) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(95/100)(1/100)}{(95/100)(1/100) + (5/100)(90/100)} = && (7)\\
&= \frac{95}{95 + 450} = && (8)\\
&= \frac{19}{109} && (9)
\end{aligned}
$$